Review: Sequences, Sets and Derivatives

Introductory Mathematical Economics

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Outline

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Research Interests

- Financial Econometrics
- Energy Finance
- Game Theory

Office Hour

Every Tuesday,
Time: 6:00pm - 7:00pm.
Room: Nangying 109

You can also send me an email: okorie.davidiheke@gmail.com

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Session Materials

At least, a day before our session, the updated slides would be available at my website. Navigate to Teaching Materials page to download the slides.

Session Rules

You must:

- not use your phone(s) during the classes.
- ask any question(s) bothering you about the on-going topic.

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Remark:

What is/are the difference(s) between Sets and Sequences?

A sequence in $\mathbb{R}^n$ is an **infinite** set of points x_k where $x_k \epsilon \mathbb{R}^n$ for each integer $k = \{1, 2, 3, 4, ...\}$

Examples in $\mathbb{R}^1$

- {1,0,1,0,0,1, ...}
- {1,2,3,4,5, ...}
- $x_k = 1 - \frac{1}{k}$ for $k = 1, 2, 3, ...$
- what of $\mathbb{R}^2$, $\mathbb{R}^3$, e.t.c. ?

Remarks:

1. A sequence converges to a limit. That is to say $d(x_k, x) \to 0$ as $k \to \infty$. The limit here is? i.e. $lim_{k\to\infty} x_k =?$
2. A sequence is monotone increasing if $x_{k+1} \geq x_k$ and monotone decreasing if $x_{k+1} \leq x_k$
3. A sequence/set is bounded by a and b if $\exists$ a, b ϵ $\mathbb{R}$ such that a $\leq x_k \leq$ b $\forall$ k.
4. Every monotone and bounded sequence converges.

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An open ball or neighbourhood with centre x and radius r is defined as B(x,r) = $\{y \in \mathbb{R}^n |\ d(x,y) < r\}$.

Open and closed sets

1. A set S $\subseteq \mathbb{R}^n$ is open iff $\forall$ x ϵ S, $\exists$ **any** radius, r > 0, such that B(x,r) ϵ S. Hence, for each x ϵ S, there is an open ball around x that is contained entirely in S
2. A set S $\subseteq \mathbb{R}^n$ is closed iff $\forall$ x ϵ S, $x_k \to x$ and x ϵ S. Hence, a closed set contains its limit points.

Egs. Sketch the first two:
1. $\{(x_1, x_2) \,\epsilon\, \mathbb{R}^2 |\ a_1 < x_1 < a_2, a_3 < x_2 < a_4\}$
2. $\{(A, B) \,\epsilon\, \mathbb{R}^2 |\ 0 \leq A \leq 7, -3 \leq B \leq 5\}$
3. Which is open and which is closed?
4. is [2,8) open or closed ?

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Bounded Set

A set S $\subset \mathbb{R}^n$ is bounded if $\exists$ 0 < r < ∞ (i.e. r is finite/defined) such that S $\subset$ B(0,r) is defined/exist. Hence, the ball completely contains S for any finite radius otherwise, it's an unbounded set.

Egs. Find the radius, r, that makes the following bounded sets

1. $S = [0, 2]$
2. $S = [2, 5]$
3. $S = [-2, 2]$
4. $S = \{0, 10, 20, ...\}$

Upper Bound

Given $A \subset \mathbb{R}$, $u \subset \mathbb{R}$ is an upper bound of A if $u \geq a \; \forall$ A. U(A) is the set for all upper bonds of A.

Lower Bound

Given $A \subset \mathbb{R}$, $l \subset \mathbb{R}$ is a lower bound of A if $l \leq a \; \forall$ A. L(A) is the set for all lower bonds of A.

Supremum

This is the least upper bound. $\sup(A) \leq u \; \forall \; u \in U(A)$. sup(A) is unique. sup(A) can be ∞ (not well defined) if A is not bounded above

Infimum

This is the highest lower bound. $\inf(A) \geq l \; \forall \; l \in L(A)$. sup(A) is unique. sup(A) can be $-\infty$ (not well defined) if A is not bounded above

Remarks

- A closed or open set is only but a necessary (and not sufficient) condition for a Bounded set and vice versa.
- Hence, not all closed sets are bounded. E.g. $\mathbb{Z}$, $\mathbb{R}$, $[-2, \infty)$, $(-\infty, 9]$, $(-\infty, \infty)$ e.t.c. are closed but unbounded.
- A set is closed (open) if its compliment is open (closed). An empty set is an open set.
- Conversely, not all bounded sets are closed. E.g. (-3,6), (2,19), e.t.c.
- However, economists are very happy when a set is closed and bounded. Why?

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Compact Set

A set S $\subset \mathbb{R}^n$ is compact if every sequence in S contains a convergent subsequence. That is, set S $\subset \mathbb{R}^n$ is compact iff it is closed and bounded.

Weierstrass Theorem:

Let D $\subset \mathbb{R}$ be **compact** and let $f : D \rightarrow \mathbb{R}$ be a **continuous function** on D, then $\exists$ points z_1 and z_2 in D such that $f(z_1) \geq f(z) \geq f(z_2)$, z ϵ D. Hence, f attains a maximum and a minimum. What are the maximum and minimum points?

Examples:

Determine the Max, Min, Infimum & Supremum of the following:

1).

$$D = (-1, 1), f(x) = x^2$$

2).

$$D = [0, 1], f(x) = \begin{cases} x, & \text{if } x = \frac{1}{n}, n = 1, 2, 3, ..., \\ 1, & \text{otherwise} \end{cases}$$

3).

$$D = \mathbb{R}, f(x) = -|x|$$

4).

$$D = \mathbb{R}, f(x) = |x|$$

5). What **assumptions** do we need to guarantee that a solution exists?

$$\max_{C} U(C)$$

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Domain and Range

Domain of f is the set of numbers, x, at which $f(x)$ is defined. What is Range?

Differentiability

A differentiable function f is differentiable at every point x_0 in its domain, D. i.e. the curve of f is smooth.

Continuity

A continuous function, f, for any sequence $\{x_n\}$ which converges to x_0 in the domain, D; $f(x_n)$ converges to $f(x_0)$. i.e. there are no breaks in the graph.

Continuously Differentiable function (C^1)

A function f is continuously differentiable if $f'(x)$ is continuous.

Twice Continuously Differentiable function (C^2)

A function f is twice continuously differentiable if $f''(x)$ is continuous.

Derivative

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Second Derivative

$$f''(x) = \frac{d^2 f(x)}{dx^2} = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Given a convex set $\mathbf{D}$, $\forall$ $\mathbf{x}$, $\mathbf{x}_0 \epsilon$ $\mathbf{D}$ and f is concave/convex then,

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \bigtriangledown f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

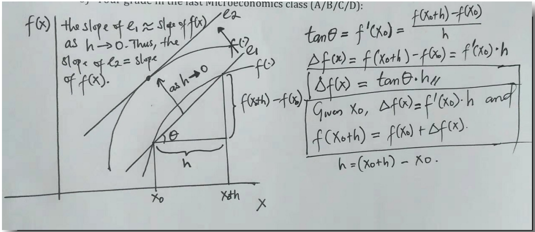

Figure 1: Derivatives and Approximations

Examples

- $f(x) = \sqrt{x}$, evaluate $f(101)$ given that $\theta = 2.86$
- $f(z) = z^4$, find $f(9.9)$

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Some Rules of Differentiation

- First Principle approach
- $D(x^k) = d(x^k) = (x^k)\prime = kx^{k-1}$, called Power Rule
- $D(\alpha) = 0$ when $\alpha \epsilon \mathbb{R}$ why?
- $(f \pm g)\prime(x) = f\prime(x) \pm g\prime(x)$, called Sum & Difference Rule
- $(f \bullet g)\prime(x) = f\prime(x)g(x) + f(x)g\prime(x)$, called Product Rule
- $(\frac{f}{g})\prime(x) = \frac{f\prime(x)g(x) - f(x)g\prime(x)}{(g(x))^2}$, called Quotient Rule
- $Df(g(x)) = f\prime(g(x))g\prime(x)$, called Chain Rule
- $D(log_a x) = \frac{1}{xlna}$, called Logarithmic Rule
- $Df(y, x)$ and $y = f(x) \rightarrow \frac{-Df_x(y,x)}{Df_y(y,x)}$, called Implicit Rule

Examples

- $f(w) = 7w^4 - 8w^2 + 9w$ find $f\prime(\mathrm{w})$,$f\prime(1)$,$f\prime\prime(\mathrm{w})$ & $f\prime\prime(\mathrm{w} = 2)$
- $f(x) = (1 - x^3)^5$, find $f\prime$
- $k = (\frac{x-1}{x+3})^{\frac{1}{3}}$ find $\frac{dk}{dx}$
- $y = (x - 3)(x^2 + 8)^3$ find $y\prime$
- $V = (mp^k)(r - p)$ find V_p
- $Y = Ak^{\alpha}l^{\beta}$ find Y_k,Y_l,Y_{kl}

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Consider $f(\mathbf{x}) = f(x_1, x_2, ..., x_k, ..., x_n)$ where x_i can vary without affecting others. i.e. x_i changes by Δx_i while other x's remain unchanged, y will change by Δy.

Definition

$$\frac{\delta f}{\delta x_k} = \lim_{h \to 0} \frac{f(x_1, x_2, ..., x_k + h, ..., x_n) - f(x_1, x_2, ..., x_k, ..., x_n)}{h}$$

Examples

- $f(x, y) = 8xy - x^3 y + xy^5$ find f_1, f_2, f_{21}, and $f_1(2, 4)$
- $g(k, m) = 75k^4$ find f_1 and f_m
- $g(s + q) = \alpha(s + q)^\beta - ms^{1-\alpha} + pq^{r+1}$, find $g\prime$, g_s and g_q
- What did you observe?

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Consider $f(\mathbf{x}) = f(x_1, x_2, ..., x_k, ..., x_n)$ where all x_i change simultaneously . i.e. all x_i change by Δx_i, y will change by Δy, total change (dy).

Definition

$$\frac{df}{d\mathbf{x}} = \lim_{\Delta \to 0} \frac{f(x_i + \Delta) - f(x_i)}{\Delta} =$$

$$\lim_{\Delta \to 0} \frac{f(x_1 + \Delta x_1, x_2 + \Delta x_2, ..., x_k + \Delta x_k, ..., x_n + \Delta x_n) - f(x_1, x_2, ..., x_k, ..., x_n)}{\Delta}$$

where Δ is a vector.

Therefore

$$dF = \frac{\delta F}{\delta x_1} dx_1 + \frac{\delta F}{\delta x_2} dx_2 + \frac{\delta F}{\delta x_k} dx_k + ... + \frac{\delta F}{\delta x_n} dx_n$$

How can we derive partial derivatives from total derivatives?

Examples

- $m = x^6 lny$, find dm
- $y = 5p^3qr + rpq - 9r^2$, find dy
- ...

Q&A Session