Economic Dynamics

Introductory Mathematical Economics

David Ihekereleome Okorie
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Motivation

Dynamic Interests

Often, we seek to not only optimize a problem at a particular point in time but over time. For example, you do not only want to maximize your consumption today and not tomorrow or next week. You would love to maximize or smoothen your consumption as long as you live. Such optimization problems are similar to statically optimizing at each point in time. However, we could easily and more generally do the same optimization in a dynamic model setup.

Time Domain

Optimization over time domain could be Discrete Time Domain, X_t, or Continuous Time Domain, $X(t)$. The choice of Time domain could depend on the modelling objective(s), algorithm complexity, matching theoretical results with empirical results et cetera.

Definitions

Discrete Time Domain is a mapping from the Discrete time set to Natural Numbers while Continuous Time Domain is a mapping from the Continuous time set to Real Line/Numbers.

Equation Preferences

Continuous time models are modelled with Differential Equations (DTE) while Discrete time models are modelled with Difference Equations (DCE).

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In a nutshell

Definition

Generally, DTE of $y(t)$ is expressed as:

$$F(t, y(t), y\prime(t), y\prime\prime(t), ..., y^{(n)}(t)) = 0$$

where

$$y\prime(t) = \frac{dy(t)}{dt}, \quad y\prime\prime(t) = \frac{d^2y(t)}{dt^2}, \quad \dots, y^{(n)}(t) = \frac{d^{(n)}y(t)}{dt^{(n)}}$$

The order of a DTE is the highest order of derivative n.

A first order DTE for $y(t)$ is expressed as:

$$\frac{dy(t)}{dt} + ay(t) = b$$

- When $a, b \neq 0$

$$\frac{dy(t)}{dt} + ay(t) = b$$

- When $a \neq 0$ and $b = 0$

$$\frac{dy(t)}{dt} + ay(t) = 0$$

- When $a = 0$ and $b \neq 0$

$$\frac{dy(t)}{dt} = b$$

- When $a = b = 0$, **what happens ?**

Remarks

1.)

$$\int ayx^n dx = \frac{ay}{n+1}x^{n+1} + c$$

2.)

$$\int \frac{1}{x} dx = lnx + c$$

3.) $lnx = 8.$ find x

4.) $e^x = 3.7.$ find x

Steps

5.) Collect like-terms

6.) Integrate out $y(t)$

7.) Apply initial conditions (i.e. t = 0) on the general solution to get the definite solution.

When $a = 0$ and $b \neq 0$

a.)

$$\frac{dy(t)}{dt} = b$$

$$\int \frac{dy(t)}{dt} dt = \int b dt$$

$$\mathbf{y(t) = bt + c}$$

$$y(t=0) = b(0) + c \rightarrow c = y(0)$$

$$\mathbf{y(t) = bt + y(0)}$$

b.)

$$\frac{dy(t)}{dt} = b$$

$$dy(t) = bdt$$

$$\int dy(t) = \int 1 dy(t) = \int bdt$$

$$\mathbf{y(t) = bt + c}$$

$$\mathbf{y(t) = bt + y(0)}$$

When $a \neq 0$ and $b = 0$

$$\frac{dy(t)}{dt} + ay(t) = 0$$

$$\frac{dy(t)}{dt} = -ay(t)$$

$$\frac{1}{y(t)}\frac{dy(t)}{dt} = -a$$

$$\int \frac{1}{y(t)}\frac{dy(t)}{dt}dt = \int -adt$$

$$\int \frac{1}{y(t)}dy(t) = \int -adt$$

$$lny(t) = -at + c$$

$$\mathbf{y(t) = exp(c)exp(\text{-}at)}$$

$$y(t=0) = exp(c)exp(-a \times 0) \rightarrow exp(c) = y(0)$$

$$\mathbf{y(t) = y(0)exp(\text{-}at)}$$

When $a, b \neq 0$

$$\frac{dy(t)}{dt} + ay(t) = b$$

if $b = 0$ (homogenous case) we have:

$$y(t) = y(0)exp(-at)$$

if $\frac{dy(t)}{dt} = 0$, we have:

$$ay(t) = b \rightarrow y(t) = \frac{b}{a}$$

$$\mathbf{y(t) = y(0)exp(\text{-}at)} + \frac{\mathbf{b}}{\mathbf{a}}$$

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Definition

A steady state is an equilibrium point where

$$\frac{dy(t)}{dt} = 0$$

Local Steady State

A steady state $y^\star(t)$ is locally stable if $\forall \; y(t) \; \epsilon \; N(y^\star(t))$ such that $y(t) \rightarrow y^\star(t)$. $N(y^\star(t)) = D(\epsilon, Y^\star(t)), \epsilon > 0$ is a neighbourhood of $y^\star(t)$

Global Steady State

A steady state $y^\star(t)$ is globally stable if $\forall \; y(t)$, $y(t) \rightarrow y^\star(t)$.

Initial values/positions tend to affect local steady states unlike global steady states.

Qualitative Analysis Examples

Solve the following DTEs and show the steady state convergence on $\frac{dy(t)}{dt}$ and $y(t)$. Assume; $y(0) = 10$, and $y\prime(0) = 2$.

1.)

$$\frac{dy(t)}{dt} - 4 = 0$$

2.)

$$\frac{d^2y(t)}{dt^2} - 4 = 0$$

3.)

$$\frac{dy(t)}{dt} - 5y(t) = 0$$

4.)

$$\frac{dy(t)}{dt} + 3y(t) = 15$$

5.)

$$\frac{dy(t)}{dt} - 2 = 9y(t)$$

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Let's take for example:

$$A = \begin{bmatrix} 2 & -7 \\ 3 & -8 \end{bmatrix}$$

The eigenvalues are values that satisfy $|A - \lambda I_n| = 0$

$$I_{n=2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\; \lambda I_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \;\; and \;\; A - \lambda I_2 = \begin{bmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{bmatrix}$$

Therefore,
a.)

$$|A - \lambda I_n| = 0 \;\; \rightarrow \;\; \begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-8-\lambda)+21 = 0 \;\; \rightarrow \;\; \lambda^2+6\lambda+5 = 0 \;\; \rightarrow \;\; (\lambda+1)(\lambda+5) = 0$$

$$\lambda = -1 \;\; or \;\; -5$$

b.) we can use the relationship below:

$$\lambda^2 - trace(A)\lambda + det(A) = 0$$

$$trace(A) = 2 - 8 = -6 \;\; |A| = \begin{vmatrix} 2 & -7 \\ 3 & -8 \end{vmatrix} = 5 \;\; \rightarrow \lambda^2 + 6\lambda + 5 = 0$$

$$\lambda = -1 \; or \; -5$$

Next, we consider how to calculate the eigenvalues of a 3-equations and 3-variables (3by3) system of linear equations.

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Let's take for example:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The eigenvalues are values that satisfy $|A - \lambda I_n| = 0$

$$I_{n=3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \lambda I_3 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \; A - \lambda I_3 = \begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

Therefore,
a.)

$$|A - \lambda I_n| = 0 \;\rightarrow\; \begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)^2 - 1] = 0 \;\rightarrow\; \lambda = 2 \;\; or \;\; 2 \;\; or \;\; 0$$

b.) we can use the relationship below:

$$\lambda^3 - trace(A)\lambda^2 + SAPM(A)\lambda - |A| = 0$$

Where SAPM is the Sum of Arbitrary Principal Minors of order $(n-1)$

$$trace(A) = 2+1+1 = 4 \;\; SLPM = 0+2+2 = 4 \;\; |A| = 0$$

$$\lambda^3 - 4\lambda^2 + 4\lambda = (2-\lambda)[(1-\lambda)^2 - 1]$$

$$\lambda = 0 \; or \; 2 \;\; or \; 2$$

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Remarks

1. Isoclines divide the (x,y) plane into sectors.
2. Intersections of isoclines are steady states.
3. Two opposing forces on an object results to a diagonal movement.

Adaptive Harvest

$$s' = f(s) - h$$

$$h' = \alpha[ph - C(h, s)]$$

Where s: resource stock, h: harvest, p: price, C(h,s): cost. $f(s)$ is concave and the cost function increases in h and decreases in s.

Isocline of $s' = 0$

From $s' = f(s) - h$ we set $s' = 0$ and plot $h = f(s)$ using the fact that $f(s)$ is concave in s. Consider what happens to s' when h $\uparrow$ and $\downarrow$.
1.)From $s' = 0$ upwards, h increases and s' decreases from $s' = f(s) - h$
2.)From $s' = 0$ downwards, h decreases and s' increases from $s' = f(s) - h$

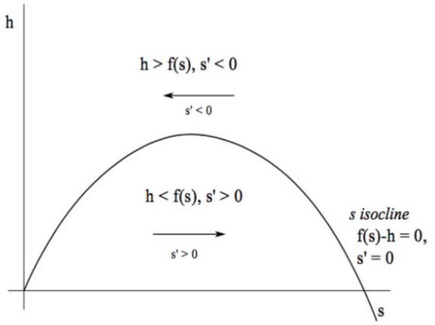

Figure 1: Isocline of Resource Stock

Isocline of $h' = 0$

From $h' = \alpha[ph - C(h,s)]$ we set $h' = 0$ and plot $h = \frac{1}{p}C(h,s)$ using the fact that $\uparrow s \rightarrow \downarrow C \rightarrow \downarrow p \rightarrow \uparrow \frac{1}{p}$. If $\Delta s, \Delta p \rightarrow 0$, then $\frac{1}{p} \rightarrow \infty$. Consider what happens to h' when s $\uparrow$ and $\downarrow$.

1.)From $h' = 0$ rightward, s increases and h' increases

2.) and Vice versa from $h' = \alpha[ph - C(h,s)]$.

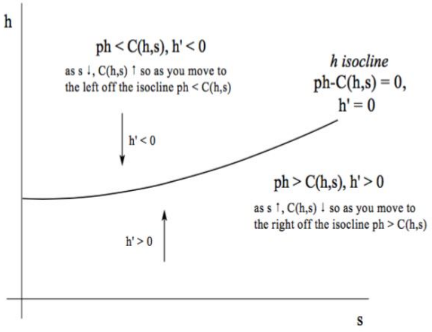

Figure 2: Isocline of Harvest

Combining the Isoclines

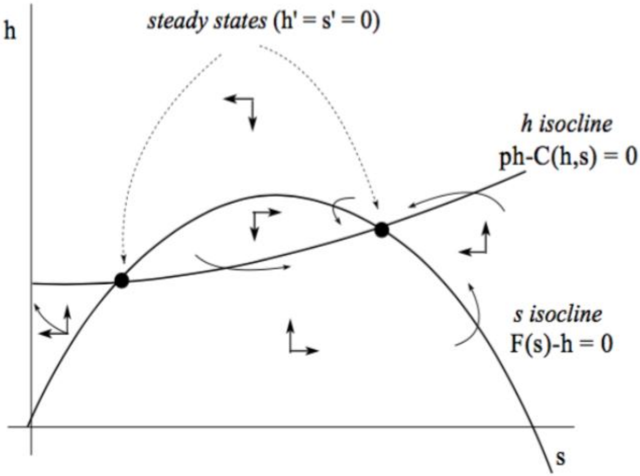

Figure 3: Phase Diagram

Try this

$$s' = f(s) - h$$

$$h' = ph - s$$

Draw the phase Diagram given that $f(s)$ is concave.