Comparative Statics

Introductory Mathematical Economics

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Outline

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Conditions

- An equilibrium system i.e. $f(x, \theta) = 0$
- Existence of a functional relationship between arguments i.e. $x = g(\theta)$

Then, we could rewrite $f(x, \theta) = 0$ as $f(g(\theta), \theta) = 0$. Differentiating w.r.t θ gives $f_1(g(\theta), \theta)g_\theta(\theta) + f_2(g(\theta), \theta) = 0$. If $f_1(g(\theta), \theta) \neq 0$ then $\frac{\delta x}{\delta \theta} = g_\theta(\theta) = \frac{-f_2(g(\theta),\theta)}{f_1(g(\theta),\theta)}$.
Generally, for $f : S \subset \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be c^1 on an open set S. Then,

$$Dg(\theta) = -\frac{Df_\theta(x, \theta)}{Df_x(x, \theta)} \, and \, |Df_x(x, \theta)| \neq 0$$

Outline

System of Equations

Conditions

- An equilibrium system i.e. $\mathbf{f}(\mathbf{X}, \Theta) = \mathbf{0}$ is a FONCs vectors.
- Existence of a functional relationship between the choice variables and the model parameters i.e. $\mathbf{X} = \mathbf{g}(\Theta)$

Then,

$$\mathbf{Dg}(\Theta) = -\frac{\mathbf{Df}_{\Theta}(\mathbf{X}, \Theta)}{\mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta)} \, and, |\mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta)| \neq 0$$

Let's say a model's choice variable set is $\mathbf{X} = (A, B, C, D)$ with the parameter set, $\Theta = (a, b, c)$. Let's further assume that $Q_1 = FOD(A)$, $Q_2 = FOD(B)$, $Q_3 = FOD(C)$, and $Q_4 = FOD(D)$.

$$\mathbf{Dg}(\Theta) = \begin{bmatrix} \frac{\delta A}{\delta a} & \frac{\delta A}{\delta b} & \frac{\delta A}{\delta c} \\ \frac{\delta B}{\delta a} & \frac{\delta B}{\delta b} & \frac{\delta B}{\delta c} \\ \frac{\delta C}{\delta a} & \frac{\delta C}{\delta b} & \frac{\delta C}{\delta c} \\ \frac{\delta D}{\delta a} & \frac{\delta D}{\delta b} & \frac{\delta D}{\delta c} \end{bmatrix} \quad \mathbf{Df}_{\Theta}(\mathbf{X}, \Theta) = \begin{bmatrix} \frac{\delta Q_1}{\delta a} & \frac{\delta Q_1}{\delta b} & \frac{\delta Q_1}{\delta c} \\ \frac{\delta Q_2}{\delta a} & \frac{\delta Q_2}{\delta b} & \frac{\delta Q_2}{\delta c} \\ \frac{\delta Q_3}{\delta a} & \frac{\delta Q_3}{\delta b} & \frac{\delta Q_3}{\delta c} \\ \frac{\delta Q_4}{\delta a} & \frac{\delta Q_4}{\delta b} & \frac{\delta Q_4}{\delta c} \end{bmatrix}$$

$$and \quad \mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta) = \begin{bmatrix} \frac{\delta Q_1}{\delta A} & \frac{\delta Q_1}{\delta B} & \frac{\delta Q_1}{\delta C} & \frac{\delta Q_1}{\delta D} \\ \frac{\delta Q_2}{\delta A} & \frac{\delta Q_2}{\delta B} & \frac{\delta Q_2}{\delta C} & \frac{\delta Q_2}{\delta D} \\ \frac{\delta Q_3}{\delta A} & \frac{\delta Q_3}{\delta B} & \frac{\delta Q_3}{\delta C} & \frac{\delta Q_3}{\delta D} \\ \frac{\delta Q_4}{\delta A} & \frac{\delta Q_4}{\delta B} & \frac{\delta Q_4}{\delta C} & \frac{\delta Q_4}{\delta D} \end{bmatrix}$$

Outline

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Every firm wants to maximize profit (TR - TC) by choosing the optimal quantity, Q, given the market prices; p, r, and w. To produce Q, firms require capital (K) and labour hour (L), hence Q = f(K,L). We set up the optimization problem for a firm as it decides the levels of capital and labour hour to employ.

Solution

$$\pi = TR - TC$$

$$\pi = pQ - (wL + rk)$$

$$\pi = pf(K, L) - wl - rk ...eqn.(1)$$

FONCs

$$\frac{\delta \pi}{\delta L} = pf_L(K, L) - w = 0 ...eqn.(2)$$

$$\frac{\delta \pi}{\delta K} = pf_k(K, L) - r = 0 ...eqn.(3)$$

Let

$$\mathbf{g} = (K, L) \quad \Theta = (p, w, r).$$

$$f_1(p, w, r) = p f_L(K, L) - w = 0$$

$$f_2(p, w, r) = p f_K(K, L) - r = 0$$

Then,

$$\mathbf{Dg}(\Theta) = -\frac{\mathbf{Df}_{\Theta}(\mathbf{X}, \Theta)}{\mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta)} \; and, |\mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta)| \neq 0$$

$$\mathbf{Dg}(\Theta) = \begin{bmatrix} \frac{\delta K}{\delta p} & \frac{\delta K}{\delta w} & \frac{\delta K}{\delta r} \\ \\ \frac{\delta L}{\delta p} & \frac{\delta L}{\delta w} & \frac{\delta L}{\delta r} \end{bmatrix} \quad \mathbf{Df}_{\Theta}(\mathbf{X}, \Theta) = \begin{bmatrix} \frac{\delta f_2}{\delta p} & \frac{\delta f_2}{\delta w} & \frac{\delta f_2}{\delta r} \\ \\ \frac{\delta f_1}{\delta p} & \frac{\delta f_1}{\delta w} & \frac{\delta f_1}{\delta r} \end{bmatrix}$$

$$and \quad \mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta) = \begin{bmatrix} \frac{\delta f_2}{\delta K} & \frac{\delta f_2}{\delta L} \\ \\ \frac{\delta f_1}{\delta K} & \frac{\delta f_1}{\delta L} \end{bmatrix}$$

$$\mathbf{Dg}(\Theta) = \begin{bmatrix} \frac{\delta K}{\delta p} & \frac{\delta K}{\delta w} & \frac{\delta K}{\delta r} \\ \\ \frac{\delta L}{\delta p} & \frac{\delta L}{\delta w} & \frac{\delta L}{\delta r} \end{bmatrix} \quad \mathbf{Df}_{\Theta}(\mathbf{X}, \Theta) = \begin{bmatrix} f_K(K,L) & 0 & -1 \\ \\ f_L(K,L) & -1 & 0 \end{bmatrix}$$

$$and \quad \mathbf{Df}_{\mathbf{X}}(\mathbf{X}, \Theta) = \begin{bmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{bmatrix}$$

$$\mathbf{Dg}(\Theta) = -\frac{\mathbf{Df}_{\Theta}(\mathbf{X},\Theta)}{\mathbf{Df}_{\mathbf{X}}(\mathbf{X},\Theta)}$$

Can also be rewritten as

$$\mathbf{Df}_{\mathbf{X}}(\mathbf{X},\Theta) \times \mathbf{Dg}(\Theta) = -\mathbf{Df}_{\Theta}(\mathbf{X},\Theta)$$

or

$$\mathbf{Dg}(\Theta) = -\mathbf{Df}_{\mathbf{X}}(\mathbf{X},\Theta)^{-1}\mathbf{Df}_{\Theta}(\mathbf{X},\Theta)$$

which is :

$$\begin{bmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{bmatrix} \times \begin{bmatrix} \frac{\delta K}{\delta p} & \frac{\delta K}{\delta w} & \frac{\delta K}{\delta r} \\ \frac{\delta L}{\delta p} & \frac{\delta L}{\delta w} & \frac{\delta L}{\delta r} \end{bmatrix} = -\begin{bmatrix} f_K(K,L) & 0 & -1 \\ f_L(K,L) & -1 & 0 \end{bmatrix}$$

Could it also be rewritten as this?

$$\begin{bmatrix} \frac{\delta K}{\delta p} & \frac{\delta K}{\delta w} & \frac{\delta K}{\delta r} \\ \frac{\delta L}{\delta p} & \frac{\delta L}{\delta w} & \frac{\delta L}{\delta r} \end{bmatrix} \times \begin{bmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{bmatrix} = -\begin{bmatrix} f_K(K,L) & 0 & -1 \\ f_L(K,L) & -1 & 0 \end{bmatrix}$$

Hence, we could examine the comparative statics collectively or individually (i.e. the matrix elements of $\mathbf{Dg}(\Theta)$). How ?
HURRAH!!!
Each element of a matrix is uniquely indexed by its row-column location. We could leverage on this knowledge.

For every comparative static indexed by **row-i**, **column-j**.
1.) Numerator is $\mathbf{Df_X}(\mathbf{X}, \Theta)$ with its **column-i** replaced with **column-j** of $\mathbf{Df}_\Theta(\mathbf{X}, \Theta)$
2.) Denominator remains $\mathbf{Df_X}(\mathbf{X}, \Theta)$
Note; $\mathbf{Df_X}(\mathbf{X}, \Theta) \neq 0$

From

$$\mathbf{Dg}(\Theta) = -\frac{\mathbf{Df}_{\Theta}(\mathbf{X},\Theta)}{\mathbf{Df}_{\mathbf{X}}(\mathbf{X},\Theta)}$$

We have

$$\begin{bmatrix} \frac{\delta K}{\delta p} & \frac{\delta K}{\delta w} & \frac{\delta K}{\delta r} \\ \frac{\delta L}{\delta p} & \frac{\delta L}{\delta w} & \frac{\delta L}{\delta r} \end{bmatrix} = -\frac{\begin{bmatrix} f_K(K,L) & 0 & -1 \\ f_L(K,L) & -1 & 0 \end{bmatrix}}{\begin{bmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{bmatrix}} ...eqn.(4)$$

Therefore,

$$\frac{\delta K}{\delta p} = \frac{-\begin{vmatrix} f_K(K,L) & pf_{KL}(K,L) \\ f_L(K,L) & pf_{LL}(K,L) \end{vmatrix}}{\begin{vmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{vmatrix}} \quad \frac{\delta K}{\delta w} = \frac{-\begin{vmatrix} 0 & pf_{KL}(K,L) \\ -1 & pf_{LL}(K,L) \end{vmatrix}}{\begin{vmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{vmatrix}}$$

$$\frac{\delta K}{\delta r} = \frac{-\begin{vmatrix} -1 & pf_{KL}(K,L) \\ 0 & pf_{LL}(K,L) \end{vmatrix}}{\begin{vmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{vmatrix}} \quad \frac{\delta L}{\delta p} = \frac{-\begin{vmatrix} pf_{KK}(K,L) & f_K(K,L) \\ pf_{LK}(K,L) & f_L(K,L) \end{vmatrix}}{\begin{vmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{vmatrix}}$$

$$\frac{\delta L}{\delta w} = \frac{-\begin{vmatrix} pf_{KK}(K,L) & 0 \\ pf_{LK}(K,L) & -1 \end{vmatrix}}{\begin{vmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{vmatrix}} \quad \frac{\delta L}{\delta r} = \frac{-\begin{vmatrix} pf_{KK}(K,L) & -1 \\ pf_{LK}(K,L) & 0 \end{vmatrix}}{\begin{vmatrix} pf_{KK}(K,L) & pf_{KL}(K,L) \\ pf_{LK}(K,L) & pf_{LL}(K,L) \end{vmatrix}}$$

Next is to calculate the determinants of the matrices. However, we must first guarantee that $|\mathbf{Df_X}(\mathbf{X}, \Theta)| \neq 0$. To calculate the determinants, any of these methods for calculating determinants;**Sarules, Laplace, Row-echelon form or Jordan form, etc.**, could be applied.

$$|\mathbf{Df_X}(\mathbf{X}, \Theta)| = p^2(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2) > 0$$

Given $f_{KK}(K,L)f_{LL}(K,L) \neq (f_{KL}(K,L))^2 \rightarrow |\mathbf{Df_X}(\mathbf{X}, \Theta)| \neq 0$

conclusions:

$$\frac{\delta K}{\delta p} = \frac{(f_L f_{KL} - f_K f_{LL})}{p(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2)} \; depends\, on\; f_L \,\&\, f_{KL}$$

$$\frac{\delta K}{\delta w} = \frac{-f_{KL}}{p(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2)} \; depends\ on\ f_{KL}$$

$$\frac{\delta K}{\delta r} = \frac{f_{LL}}{p(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2)} < 0$$

$$\frac{\delta L}{\delta p} = \frac{(f_K f_{LK} - f_L f_{KK})}{p(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2)} \textit{ depends on } f_K \,\&\, f_{LK}$$

$$\frac{\delta L}{\delta w} = \frac{f_{KK}}{p(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2)} < 0$$

$$\frac{\delta L}{\delta r} = \frac{-f_{LK}}{p(f_{KK}(K,L)f_{LL}(K,L) - (f_{KL}(K,L))^2)} \textit{ depends on } f_{LK}$$

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Sherwin Rosens model of the Irish potato famine - Journal of Political Economy (1999)

- The Irish potato famine has frequently been cited as a historical example of a Giffen good, a good whose demand is increasing in price.
- Rosen argues that this interpretation is not correct and that the observed data is due to the nature of potato farming where part of this year's crop must be saved to plant next year's.

Potatoes are important in the diets of many poor people, but much less important in the diets of wealthy people. When the cheap good comprises a large share of expenditures, an increase in price crowds out spending on more expensive goods and results in an increase in quantity demanded for the cheap good. However, potatoes are capital goods as well as consumption goods. Next year's crop is produced by planting potato buds (eyes) obtained from the current crop by withholding a portion of the crop from consumption.

Seed potatoes as % of crop

- Russia - 25%
- USA - 7.9%
- Ireland during the period of potato famine - 15%
  - Great Irish Famine (1845-1847) - Precipitated by the appearance of a fungus on the Irish potato crop.

Irish law and social tradition provided for subdivision of land such that all sons inherited equal shares in a farm. Prospect of inheriting land encourage males to marry young, which means larger families and more subdivision. In 1845, 24% of all Irish tenant farms were of 0.4 to 2 hectares (one to five acres) in size, while 40% were of two to six hectares (five to fifteen acres). Plots became so small that only one crop, potatoes, could be grown in sufficient amounts to feed a family. In Ireland, future years' crops were seeded by simply leaving some of the potatoes unharvested in the ground.

- Variables:
  - s: seed crop (potatoes used for seed)
  - p: price of potatoes
- Parameters:
  - g: reproduction rate of potato
  - r: interest rate
- Functions:
  - K(s): marginalcost of planting and storage
  - C(p): consumption demand for potatoes
- Assumptions:
  - $\frac{dK}{ds} > 0$
  - $\frac{d^2K}{ds} \geq 0$
  - $g > r$

At equilibrium:

$$s - \frac{C(p)}{g} = 0$$

At the steady state condition, the growth in the seed crop equals consumption demand.

$$\frac{pg}{1+r} = \frac{pr}{1+r} + K(s)$$

Price times the internal rate of return from growing potatoes equals the marginal cost of planting and storage.

a) Write down $Dg(\theta)$

Let's define $\mathbf{g} = (s, p)$ and $\theta = (g, r)$. It follows that $s = f_1(g, r)$ and $p = f_2(g, r)$

$$D\mathbf{g}(\theta) = \begin{bmatrix} \frac{\delta s}{\delta g} & \frac{\delta s}{\delta r} \\ \\ \frac{\delta p}{\delta g} & \frac{\delta p}{\delta r} \end{bmatrix}$$

b) Write down $Df_\theta(\mathbf{g}, \theta)$

$$Df_\theta(\mathbf{g}, \theta) = \begin{bmatrix} \frac{\delta f_1}{\delta g} & \frac{\delta f_1}{\delta r} \\ \frac{\delta f_2}{\delta g} & \frac{\delta f_2}{\delta r} \end{bmatrix} = \begin{bmatrix} \frac{C(p)}{g^2} & 0 \\ \frac{p}{1+r} & \frac{-p(1+g)}{(1+r)^2} \end{bmatrix}$$

c) Write down $Df_\mathbf{g}(\mathbf{g}, \theta)$

$$Df_\mathbf{g}(\mathbf{g}, \theta) = \begin{bmatrix} \frac{\delta f_1}{\delta s} & \frac{\delta f_1}{\delta p} \\ \frac{\delta f_2}{\delta s} & \frac{\delta f_2}{\delta p} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-C_p(p)}{g} \\ -K_s(s) & \frac{(g-r)}{(1+r)} \end{bmatrix}$$

we also need to show that

$$|Df_\mathbf{g}(\mathbf{g}, \theta)| = \frac{(g-r)}{(1+r)} - \frac{K_s(s)C_p(p)}{g} > 0 \; Since \; C_p(p) < 0\,; def.$$

d) Write down the Implicit Function Theorem

$$D\mathbf{g}(\theta) = -(Df_{\mathbf{g}}(\mathbf{g},\theta))^{-1} \times Df_{\theta}(\mathbf{g},\theta)$$

$$\begin{bmatrix} \frac{\delta s}{\delta g} & \frac{\delta s}{\delta r} \\ \frac{\delta p}{\delta g} & \frac{\delta p}{\delta r} \end{bmatrix} = - \begin{bmatrix} 1 & \frac{-C_p(p)}{g} \\ -K_s(s) & \frac{(g-r)}{(1+r)} \end{bmatrix}^{-1} \times \begin{bmatrix} \frac{C(p)}{g^2} & 0 \\ \frac{p}{1+r} & \frac{-p(1+g)}{(1+r)^2} \end{bmatrix}$$

e) Write down $\frac{\delta p}{\delta g}$ and determine the sign

$$\frac{\delta p}{\delta g} = \frac{-\begin{vmatrix} 1 & \frac{C(p)}{g^2} \\ -K_s(s) & \frac{p}{1+r} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{-C_p(p)}{g} \\ -K_s(s) & \frac{g-r}{1+r} \end{vmatrix}} < 0$$

f) Write down $\frac{\delta s}{\delta g}$

$$\frac{\delta s}{\delta g} = \frac{-\begin{vmatrix} \frac{C(p)}{g^2} & \frac{-C_p(p)}{g} \\ \frac{p}{1+r} & \frac{g-r}{1+r} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{-C_p(p)}{g} \\ -K_s(s) & \frac{g-r}{1+r} \end{vmatrix}}$$

g) How to determine the sign of $\frac{\delta s}{\delta g}$ Taking advantage of the elasticity of demand

Since the denominator is greater than zero, our interest is on the sign of the numerator, say N.

$$N = -(\frac{C(p)}{g^2} \times \frac{g-r}{1+r} + \frac{C_p(p)}{g} \times \frac{p}{1+r})$$

- $N > 0$ iff $|\frac{C_p(p)}{g} \times \frac{p}{1+r}| > \frac{C(p)}{g^2} \times \frac{g-r}{1+r} \rightarrow |\epsilon_p| > |\frac{g-r}{g}|$
- $N < 0$ iff $|\frac{C_p(p)}{g} \times \frac{p}{1+r}| < \frac{C(p)}{g^2} \times \frac{g-r}{1+r} \rightarrow |\epsilon_p| < |\frac{g-r}{g}|$
- $N = 0$ iff $|\frac{C_p(p)}{g} \times \frac{p}{1+r}| = \frac{C(p)}{g^2} \times \frac{g-r}{1+r}. \rightarrow |\epsilon_p| = |\frac{g-r}{g}|$

Good Luck!!!
Midterm Examination
Details:
Date: Saturday 16th November
Time:7:10pm - 8:50pm
Classroom: Rm.102, Jimei2 Building