Inverse and Implicit Functions

Introductory Mathematical Economics

David Ihekereleome Okorie

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Outline

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Recall

It shows that

$$V_{\theta\theta}(\theta) \geq F_{\theta\theta}(x, \theta)$$

That is,

$$V_{\theta\theta}(\theta) = F_{\theta\theta}(x, \theta) + F_{x\theta}(x, \theta)X_{\theta}$$

$$Then,\ F_{x\theta}(x, \theta)X_{\theta} \geq 0.$$

$$Implying\, sign(F_{x\theta}(x, \theta)) = sign(X_{\theta})$$

Interest

At this point,our interest gears towards establishing the effect of the model parameters (θ) on the optimal solution $X^{\star}(\theta)$. Most times, the objective functions are not explicitly defined, then we have to leverage on the fact that $sign(F_{x\theta}(x, \theta)) = sign(X_{\theta})$ from $F_{x\theta}(x, \theta)X_{\theta} \geq 0$.

Show the direction of each parameter impact on the choice variable(s).

Eg.1

$$\underset{x>0}{\arg\max}\ px^{\frac{1}{2}} - wx$$

$$f_x = \frac{1}{2}px^{\frac{-1}{2}} - w, f_{xp} = \frac{1}{2\sqrt{x}}, \, and \, f_{xw} = -1$$

Therefore, $\frac{\delta x}{\delta w} < 0$ and $\frac{\delta x}{\delta p} < / > 0$

Eg.2

$$\underset{x>0,y>0}{\arg\max}\ U(x,y)\, s.t.\, p_x x + p_y y \leq I$$

$$Ł_x = U_x(x,y) - \lambda P_x, Ł_y = U_y(x,y) - \lambda P_y, Ł_{xp_x} = Ł_{yp_y} = -\lambda$$

Therefore, $\frac{\delta i}{\delta p_i} < 0$ otherwise, 0. where $i = \{x, y\}$

Eg.3: Pollution Control

x_i : Pollution abatement of firm i. $A = \sum_{i=1}^{n} x_i$: Total abatements of all firms. $c(x_i, \theta_i)$: Abatement cost of firm i. θ_i : Abatement cost parameter. $B(A)$: Social benefit of abatement.

$$f_x = B_A(\sum_{i=1}^{n} x_i) - c_x(x_i, \theta_i)\, and\, f_{x\theta} = -C_{x\theta}(x_i, \theta_i)$$

Therefore, $\frac{\delta x}{\delta \theta} < 0$

Eg.4

$$\arg\max_{x_1, x_2 \geq 0} -\theta_1 x_1 - \theta_2 x_2$$

$$f_{x_i} = -\theta_i\, and\, f_{x_i \theta_i} = -1,\, otherwise\, 0. Where\, i = \{x, y\}$$

Therefore $\frac{\delta x_i}{\delta \theta_i} < 0$ otherwise, 0.

Remarks:

Considering examples 2 & 4, we could not establish cross effects directly. We therefore need to formally apply the implicit or inverse function theorem to establish both direct and cross (indirect) effects of the model's parameters on the choice variable(s).
These approaches are used when one variable is a function of another, whether or not the functional form or relationship is explicitly defined.

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For $g : X \to \theta$ to be invertible, g must be a one-to-one correspondence (bijection) (i.e. both one-to-one (injective) and onto (surjective) function).
$\forall$ one-to-one, $f(x_1) = f(x_2)$ iff $x_1 = x_2$ but for onto function $f(x_1) = f(x_2)$ doesn't necessarily mean that $x_1 = x_2$.

Remarks:

1. A matrix $A = \{a_{i,j}\}_{n\times m}$ is invertible if $|A| \neq 0$
2. Injective but not surjective functions has no defined $f^{(-1)}$
3. Surjective but not injective functions has no well-defined $f^{(-1)}$
4. Bijection (Injective and surjective) functions has well-defined $f^{(-1)}$

For an invertible function $\theta = g(x)$, with inverse, $x = g^{-1}(\theta)$ where $g^{-1}(g(x)) = g(g^{-1}(x))$.

$$\frac{\delta x}{\delta \theta} = D_\theta g^{-1}(\theta) = (D_x g(x))^{-1}$$

Remarks:

1. For invertible functions, make the domain variable the subject of the formula.
2. Replace the Domain variable with that of the Range and vice versa.

Examples

Are these functions invertible?

$$1).\ y = 3x^2 - 2 \quad 2).\ y = lnx \quad 3).\ y = 4x - 5$$

Show that the derivative of an inverse function is the inverse of the derivative of the original function with the examples above.

Soln.

(2) and (3) are invertible while (1) is not. Why?

$$3).f^{-1}: x = \frac{1}{4}y + \frac{5}{4} \; and \; f: y = 4x - 5 \;\rightarrow\; Df^{(-1)} = (Df)^{-1} = \frac{1}{4}$$

$$2).f^{-1}: x = e^{(y)} \; and \; f: y = ln(x) \;\rightarrow\; Df^{(-1)} = (Df)^{-1} = x$$

$$1).f^{-1}: x = \sqrt{\frac{1}{3}y + \frac{2}{3}} \; and \; f: y = 3x^2 - 2 \;\rightarrow\; Df^{(-1)} = \frac{1}{\pm 6x}$$

$$\neq (Df)^{-1} = \frac{1}{6x}$$

In conclusion, for an inverse function derivative rule to hold, the function must be invertible, i.e. a well defined inverse function must exist (one to one correspondence) else, the Implicit function derivative rule applies i.e. whether the inverse function is well defined or not. We will revisit example (1) soon!

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Conditions

- An equilibrium system i.e. $f(x, \theta) = 0$
- Existence of a functional relationship between arguments i.e. $x = g(\theta)$

Then, we could rewrite $f(x, \theta) = 0$ as $f(g(\theta), \theta) = 0$. Then, differentiating w.r.t θ gives $f_1(g(\theta), \theta)g_\theta(\theta) + f_2(g(\theta), \theta) = 0$. If $f_1(g(\theta), \theta) \neq 0$ then $\frac{\delta x}{\delta \theta} = g_\theta(\theta) = \frac{-f_2(g(\theta),\theta)}{f_1(g(\theta),\theta)}$. Generally, for $f : S \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$ be c^1 on an open set S. Then,

$$Dg(\theta) = -\frac{Df_\theta(x, \theta)}{Df_x(x, \theta)} \, and, |Df_x(x, \theta)| \neq 0$$

Can we now confirm that the derivative of the previous onto function (since $y = f(x) = 0 \, \epsilon \, \mathbb{R}$) is same using the direct derivative method and implicit derivative rule?

Examples

$$Find\ \frac{\delta y}{\delta x}$$

1. $3y^2x - xy + 5 = 19$

Inverse function or Implicit theorem and why?

2.

$$Max_L PQ - wL$$

where $P = 1$ and $Q = f(L)$

Inverse function or/and Implicit theorem and why?

1.)

$$3y^2 + x(6y\frac{\delta y}{\delta x}) - (x\frac{\delta y}{\delta x} + y) = 0$$

$$\frac{\delta y}{\delta x} = \frac{1-3y}{x(6-1)}$$

2.)

Implicit Theorem:

$$Max f(L) - wL \quad \rightarrow \quad f_L(L) - w = 0 \;\; and \;\; w = g(L)$$

Since both conditions are satisfied, then:

$$\frac{\delta L}{\delta w} = \frac{1}{f_{LL}(L)}$$

...

Inverse Function Theorem:
Since the function is one-to-one correspondence, we have from (1)
$w = f_L(L)$ with an inverse function of $L = f_L^{-1}(w)$. Therefore,

$$\frac{\delta L}{\delta w} = f_{Lw}^{-1}(w) = \left(\frac{\delta w}{\delta L}\right)^{-1} = (f_{LL}(L))^{-1} = \frac{1}{f_{LL}(L)}$$

Next week, we would dive into a formal and complete system comparative static analysis uing the Implicit Derivative Rule in matrix forms.