Envelope Theorem and Applications

Introductory Mathematical Economics

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Outline

Introduction
- Motivation

Envelop Theorem
- Introduction
- Envelop Theorem with Unconstrained Optimization
- Envelop Theorem with Constrained Optimization

Examples

Outline

Introduction
Motivation

Envelop Theorem
Introduction
Envelop Theorem with Unconstrained Optimization
Envelop Theorem with Constrained Optimization

Examples

Outline

Introduction
- Motivation

Envelop Theorem
- Introduction
- Envelop Theorem with Unconstrained Optimization
- Envelop Theorem with Constrained Optimization

Examples

Motivation

At this point, we have walked through how to optimize static objective functions given (in)equality constraint(s). In addition, we have systematically analyze the processes which show that our solutions are truly local (global) maximizers/minimizers. Finally, we established the Value function.

The Task

The task now is to see how the value function respond to changes in the parameter(s) of the model i.e.

COMPARATIVE STATICS.

In Mathematical Language

$$\arg\max_{x \geq 0} F(x, \theta),\ s.t.\ g(x, \theta) = 0$$

$$let\, x^{\star}(\theta)\, \epsilon\, \arg\max_{x \geq 0} F(x, \theta),\ s.t.\ g(x, \theta) = 0$$

$$then\, V(\theta) = F(x^{\star}(\theta), \theta) \geq F(x'(\theta'), \theta),\ x^{\star} \neq x'\, given\, \theta$$

What we are looking for are $V_{\theta}(\theta)$ and $V_{\theta\theta}(\theta)$

Outline

Introduction
  Motivation

Envelop Theorem
  Introduction
  Envelop Theorem with Unconstrained Optimization
  Envelop Theorem with Constrained Optimization

Examples

Outline

Introduction
Motivation

Envelop Theorem
Introduction
Envelop Theorem with Unconstrained Optimization
Envelop Theorem with Constrained Optimization

Examples

Envelop Theorem

Envelop theorem therefore provides us with an alternative and easier approach to establish the effect of a change in the model parameter(s) on the value function for further analysis and understanding of the modelled economic phenomena.

We would then examine Envelope Theorem (ET) in the context of unconstrained optimization and constrained optimization.

Outline

Introduction
Motivation

Envelop Theorem
Introduction
Envelop Theorem with Unconstrained Optimization
Envelop Theorem with Constrained Optimization

Examples

First Order ET

It stipulates that

$$V_\theta(\theta) = F_\theta(x,\theta)|_{x=x^\star}$$

That is, We differentiate the objective function $F(x,\theta)$ and evaluate at $x^\star$.

Second Order ET

It shows that

$$V_{\theta\theta}(\theta) \geq F_{\theta\theta}(x,\theta)$$

That is,

$$V_{\theta\theta}(\theta) = F_{\theta\theta}(x,\theta) + F_{x\theta}(x,\theta)X_\theta$$

$$Then,\ F_{x\theta}(x,\theta)X_\theta \geq 0.$$

$$Implying\ that\ sign(F_{x\theta}(x,\theta)) = sign(X_\theta)$$

Proof

$$Given\, V(\theta) = F(x^\star(\theta), \theta) \geq F(x'(\theta'), \theta)$$

Note, $x^\star(\theta)$ is the optimal solution given θ and $x'(\theta')$ is the optimal solution given θ'. Recall the Taylor Series

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

The second order TS linear approximation of θ' (old) to θ (new). We have:

$$F(x'(\theta'),\theta) \approx F(x'(\theta'),\theta)|_{\theta=\theta'} + F_\theta(x'(\theta'),\theta)|_{\theta=\theta'}(\theta-\theta') + \frac{F_{\theta\theta}(x'(\theta'),\theta)|_{\theta=\theta'}}{2!}(\theta-\theta')^2$$

$$V(\theta) \approx V(\theta)|_{\theta=\theta'} + V_\theta(\theta)|_{\theta=\theta'}(\theta-\theta') + \frac{V_{\theta\theta}(\theta)|_{\theta=\theta'}}{2!}(\theta-\theta')^2$$

nth order TS approximation is an nth level approximate of the value of a function from a known point (old) to an unknown point (new). For better understanding of what is going on, consider this simple example.

Show the first-five order approximations of f(x = 3)

$$f(x) = x^3 + 2x^2 - 5$$

Given $f(x=1) = -2$

$0th$ Order

$$f(x=3) \approx f(x)|_{x=1} = (1)^3 + (1)^2 - 5 = -2$$

$1st$ Order

$$f(x=3) \approx f(x)|_{x=1} + f^1(x)|_{x=1}(3-1) = -2 + (3x^2 + 4x)|_{x=1}(2)$$

$$= -2 + 14 = 12$$

$2nd$ Order

$$f(x=3) \approx f(x)|_{x=1} + f^1(x)|_{x=1}(3-1) + \frac{f^2(x)|_{x=1}}{2!}(3-1)^2$$

$$= 12 + \frac{(6x+4)|_{x=1}}{2}(2)^2 = 12 + 20 = 32$$

$3rd$ Order

$$f(x=3) \approx f(x)|_{x=1} + f^1(x)|_{x=1}(3-1) + \frac{f^2(x)|_{x=1}}{2!}(3-1)^2$$

$$+ \frac{f^3(x)|_{x=1}}{3!}(3-1)^3 = 32 + \frac{(6)|_{x=1}}{6}(2)^3 = 32 + 8 = 40$$

$4th$ Order

$$f(x=3) \approx f(x)|_{x=1} + f^1(x)|_{x=1}(3-1) + \frac{f^2(x)|_{x=1}}{2!}(3-1)^2$$

$$+\frac{f^3(x)|_{x=1}}{3!}(3-1)^3 + \frac{f^4(x)|_{x=1}}{4!}(3-1)^4$$

$$= 40 + \frac{(0)|_{x=1}}{24}(2)^4 = 40 + 0 = 40$$

$5th$ Order

$$f(x=3) \approx f(x)|_{x=1} + f^1(x)|_{x=1}(3-1) + \frac{f^2(x)|_{x=1}}{2!}(3-1)^2$$

$$+\frac{f^3(x)|_{x=1}}{3!}(3-1)^3 + \frac{f^4(x)|_{x=1}}{4!}(3-1)^4 + \frac{f^5(x)|_{x=1}}{5!}(3-1)^5$$

$$= 40 + \frac{(0)|_{x=1}}{120}(2)^5 = 40 + 0 = 40$$

Remarks

1. The true value of $f(x = 3) = 40$
2. The $0th$ order approximation said $f(x = 3) \approx -2$ and the $3rd$ order approximation is $f(x = 3) \approx 40$
3. After the $3rd$ order approximation, subsequent approximations still gives 40.
4. Generally, higher order approximations do better job in getting the true value than lower order approximations.

Now we would continue with the Second order ET proof. This should look familiar now.The second order TS approximation of θ' (old) to θ (new). We have:

$$F(x'(\theta'), \theta) \approx F(x'(\theta'), \theta)|_{\theta=\theta'} + F_\theta(x'(\theta'), \theta)|_{\theta=\theta'}(\theta - \theta') + \frac{F_{\theta\theta}(x'(\theta'), \theta)|_{\theta=\theta'}}{2!}(\theta - \theta')^2$$

$$V(\theta) \approx V(\theta)|_{\theta=\theta'} + V_\theta(\theta)|_{\theta=\theta'}(\theta - \theta') + \frac{V_{\theta\theta}(\theta)|_{\theta=\theta'}}{2!}(\theta - \theta')^2$$

$$Given\, V(\theta) \geq F(x'(\theta'), \theta)$$

$$V(\theta)|_{\theta=\theta'} + V_\theta(\theta)|_{\theta=\theta'}(\theta-\theta') + \frac{V_{\theta\theta}(\theta)|_{\theta=\theta'}}{2!}(\theta-\theta')^2 \geq F(x'(\theta'), \theta)|_{\theta=\theta'} +$$

$$F_\theta(x'(\theta'), \theta)|_{\theta=\theta'}(\theta - \theta') + \frac{F_{\theta\theta}(x'(\theta'), \theta)|_{\theta=\theta'}}{2!}(\theta - \theta')^2$$

$$V(\theta') + V_\theta(\theta')(\theta-\theta') + \frac{V_{\theta\theta}(\theta)|_{\theta=\theta'}}{2!}(\theta-\theta')^2 \geq F(x'(\theta'), \theta') + F_\theta(x'(\theta'), \theta')$$

$$(\theta - \theta') + \frac{F_{\theta\theta}(x'(\theta'), \theta)|_{\theta=\theta'}}{2!}(\theta - \theta')^2$$

By optimality, $V(\theta') = F(x'(\theta')$ and by first order ET $V_\theta(\theta') = F_\theta(x'(\theta'), \theta')$ Then

$$V_{\theta\theta}(\theta)|_{\theta=\theta'} \geq F_{\theta\theta}(x'(\theta'), \theta)|_{\theta=\theta'} \ldots eqn.(1)$$

However, from first order ET,

$$V_\theta(\theta) = F_\theta(x(\theta), \theta)$$

Differentiate both sides w.r.t θ

$$V_{\theta\theta}(\theta) = F_{\theta x}(x(\theta), \theta)X_\theta + F_{\theta\theta}(x(\theta), \theta) \ldots eqn.(2)$$

Comparing eqn.(1) and eqn.(2), it is clear that

$$F_{\theta x}(x(\theta), \theta)X_\theta \geq 0$$

and

$$sign(F_{\theta x}(x(\theta), \theta)) = sign(X_\theta)$$

Outline

Introduction
Motivation

Envelop Theorem
Introduction
Envelop Theorem with Unconstrained Optimization
Envelop Theorem with Constrained Optimization

Examples

Consider a Household utility maximization problem

$$\arg\max_{x>0,y>0} U(x,y),\ s.t.\, p_x x + p_y y \leq I$$

Then,

$$\mathscr{L}_{max} = U(x,y) + \lambda(I - p_x x - p_y y)$$

$$let\,(x^\star(\theta), y^\star(\theta))\,\epsilon \arg\max_{x>0,y>0} U(x,y) + \lambda(I - p_x x - p_y y)$$

Where the parameter and choice variable space θ and $\mathbf{Z}$ are

$$\theta = \{p_x\ p_y\ I\} \quad \mathbf{Z} = \{x\ y\ \lambda\}$$

Then

$$V(\theta) = \mathscr{L}(x(\theta), y(\theta), \lambda(\theta), \theta) = U(x(\theta), y(\theta)) + \lambda(\theta)(I - p_x x(\theta) - p_y y(\theta)$$

$$D_\theta V(\theta) = \frac{\delta \mathscr{L}(x(\theta), y(\theta), \lambda(\theta), \theta)}{\delta\theta} = \frac{\delta\mathscr{L}}{\delta x} X_\theta + \frac{\delta\mathscr{L}}{\delta y} Y_\theta + \frac{\delta\mathscr{L}}{\delta\lambda}\lambda_\theta + \frac{\delta\mathscr{L}}{\delta\theta}$$

since $(x(\theta), y(\theta))$ is the optimal solution. From FONCs

$$\frac{\delta \mathscr{L}}{\delta x} = \frac{\delta \mathscr{L}}{\delta y} = \frac{\delta \mathscr{L}}{\delta \lambda} = 0$$

Then,

$$D_\theta V(\theta) = \frac{\delta \mathscr{L}(x(\theta), y(\theta), \lambda(\theta), \theta)}{\delta \theta} = \frac{\delta \mathscr{L}}{\delta \theta}|_{Z=Z^\star}$$

$$D_{P_x} V(\theta) = -\lambda x|_{Z=Z(\theta)} = -\lambda(\theta)x(\theta)$$

$$D_{P_y} V(\theta) = -\lambda y|_{Z=Z(\theta)} = -\lambda(\theta)y(\theta)$$

$$and \;\; D_I V(\theta) = -\lambda|_{Z=Z(\theta)} = -\lambda(\theta)$$

Outline

Introduction
Motivation

Envelop Theorem
Introduction
Envelop Theorem with Unconstrained Optimization
Envelop Theorem with Constrained Optimization

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Show the first order ET holds in both examples

Eg.1

$$\arg\max_{x>0} px^{\frac{1}{2}} - wx$$

Eg.2

$$\arg\max_{x>0,y>0} 4xy$$

s.t. $p_x x + p_y y \leq I$