Inequality Constraints and Shadow Price

Introductory Mathematical Economics

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Leading Principal Minor

A square matrix, $\{A\}_{ij}$ has n leading principal minors. Where $n = i = j$

Given that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The leading principal minors are:

$$D_1 = \begin{bmatrix} a_{11} \end{bmatrix}, D_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} and\, D_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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Arbitrary Principal Minor

A square matrix, $\{A\}_{ij}$ has $k-order$ arbitrary principal minors. Where $k=1,2,...,n$ and $n=i=j$. This is derived from cancelling different and unique equal $(n-k)$ number of rows and columns. Using the already defined $\{A\}_{ij}$.

The arbitrary principal minors are:

$$\Delta_1^1 = \begin{bmatrix} a_{11} \end{bmatrix}, \Delta_1^2 = \begin{bmatrix} a_{22} \end{bmatrix}, and\, \Delta_1^1 = \begin{bmatrix} a_{33} \end{bmatrix}$$

$$\Delta_2^1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \Delta_2^2 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, and\, \Delta_2^3 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

and

$$\Delta_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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Elementary Row Operations

These operations do not change the solution set of a matrix relative to the original system.

- Interchange two rows of a matrix
- Change a row by adding a multiple of another row to it
- Multiple each element in a row by the same nonzero scaler

Leading Zeros

A row of a matrix has k leading zeros if the first k element(s) of the row are all zeros and the $(k+1)th$ element is not zero in the same row

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Row Echelon Form

A matrix is in row echelon form if each row has more leading zeros than the row preceding it. This is also called Gaussian form. It can be obtained by elementary row operations. Note, it is different from Reduced Row Echelon Form/Gaussian Jordan Form.

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Definition; Rank

The number of nonzero rows in a matrix's row echelon form is its rank. Given $A = (a_{ij})_{m\times n}$, $Rank(A) \leq min(n, m)$. A is full rank if $Rank(A) = min(n, m)$. Alternatively, Rank(A) is the order of the largest $(n \times n)$ minor of A that is different from zero.

Examples:

Find the Rank of the following matrices. Which is full rank?

$$A = \begin{bmatrix} 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 8 & 9 \\ 3 & -1 & 0 \end{bmatrix}, C = \begin{bmatrix} 8 & 2 \\ 6 & -11 \end{bmatrix},$$

$$D = \begin{bmatrix} 8 & 2 \\ 6 & -11 \\ 1 & 0 \end{bmatrix} \; and \; E = \begin{bmatrix} 8 & 2 & 4 \\ 1 & 1 & 1 \\ 6 & -11 & 5 \end{bmatrix}$$

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Let $h_E = \{h_i(x^\star)\}_{i \epsilon E(x^\star)}$ be the set of **binding** constraints at $x^\star$, where $E(x^\star)$ denote the set of binding constraints at $x^\star$. Then, a solution exists if (CQ1) $Dh_{E(x^\star)}$ is full rank and (the CQ2 or slater's condition) $\{h_i(x^\star)\}_{i \epsilon E(x^\star)}$ are concave (pseudo-concave) and there exist $x\prime$ such that $h_i(x\prime) > 0$.

Remarks:

1. CQ1 uses the Jacobian matrix of the binding constraints
2. A square matrix $X = \{a_{ij}\}_{n \times n}$ of order n is full rank, i.e r(A) = n if $det(X) \neq 0$.
3. We have 2^k different combinations of binding constraints to consider. Where k is the # of binding constraints. $k = k, (k-1), (k-2), ..., (k-k+1), (k-k)$. i.e. all constraints binding, (k-1) constraint(s) binding, etc. Under each consideration, form the jacobian matrix (individually or collectively) and check for full rank with the optimal solution set, $\mathbf{x}$, that satisfies **all** the constraints.
4. If CQ1 fails, then those optimal solution set,$\mathbf{x}$, are additional solutions to the optimization problem.

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Example 1

$$\arg\max_{c \geq 0} U(c)$$

subject to $pc \leq I, c \geq 0$.

Soln.

1. $k = 2$ **constraints are binding**

i.e. $pc = l$ and $c = 0$

$J_{1,2} = \begin{bmatrix} p \\ 1 \end{bmatrix}$ or $J_1 = \begin{bmatrix} p \end{bmatrix}$ and $J_2 = \begin{bmatrix} 1 \end{bmatrix}$. Notice that the jacobian matrices are full rank hence, CQ1 is satisfied

2. $k = 1$ **constraint is binding**

1.1. $pc = l$ **is binding**, $J_1 = \begin{bmatrix} p \end{bmatrix}$ which is full rank

1.2 $c = 0$ **is binding**, $J_2 = \begin{bmatrix} 1 \end{bmatrix}$, full rank

3. $k = 0$ **constraint is binding**

no jacobian matrix. **Generally, CQ1 is satisfied.**

CQ2 is also satisfied since, $pc = l$ **and** $c = 0$ **are concave in c.**

Example 2

$$\underset{c_1 \geq 0, c_2 \geq 0}{\arg\max}\, U(c_1, c_2)$$

subject to $p_1 c_1 + p_2 c_2 \leq I$, $c_1 \geq 0$, and $c_2 \geq 0$

Soln.

1. $k = 3$ **constraints are binding**

i.e. $p_1 c_1 + p_2 c_2 = l$, $c_1 = 0$, and $c_2 = 0$

$J_{1,2,3} = \begin{bmatrix} p_1 & p_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $J_1 = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$, $J_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $J_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Notice that the jacobian matrices are full rank hence, CQ1 is satisfied

2. $k = 2$ **constraints are binding**

2.1. $p_1c_1 + p_2c_2 = l$ **and** $c_1 \geq 0$ **are binding**, $J_{1,2} = \begin{bmatrix} p_1 & p_2 \\ 1 & 0 \end{bmatrix}$ full rank

2.2 $p_1c_1 + p_2c_2 = l$, **and** $c_2 = 0$ **are binding**, $J_{1,3} = \begin{bmatrix} p_1 & p_2 \\ 0 & 1 \end{bmatrix}$, full rank

2.3 $c_1 = 0$, **and** $c_2 = 0$ **are binding**, $J_{2,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, full rank

3. $k = 1$ **constraint is binding**

3.1. $p_1c_1 + p_2c_2 = l$ **is binding**, $J_1 = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$ which is full rank

3.2 $c_1 = 0$ **is binding**, $J_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, full rank

3.3 $c_2 = 0$ **is binding**, $J_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, full rank

4. $k = 0$ **constraint is binding**

no jacobian matrix. **Generally, CQ1 is satisfied.**

CQ2 is also satisfied since $pc = l$ **and** $c = 0$ **are concave in** c_1 **and** c_2**.**

More examples

$$\arg\max_{x,y} xy$$

subject to $x \geq 0$, $y \geq 0$, and $(1-x)^3 - y \geq 0$

$$\arg\max_{x} x - 1$$

subject to $-(x-1)^2 \geq 0$

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Remarks:

1. Optimize using only the binding constraints.
2. Convert the inequality constraints ($\leq$) to equality constraints.
3. State the FONCs
4. State the KKT condtions
5. Solve for all the possible solutions in 2^k possible combinations of the constraints. where k is the # of binding constraints
6. In each combination, rewrite all FONCs and KKTCs to solve for the solutions and check if it satisfy all the (binding) constraints
7. Check all the possible solutions to find the optimal solution

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KKT Conditions

$\{\lambda_i(g_i(x) - b_i)\}_{i=1}^k = 0$, $\{\lambda_i\}_{i=1}^k \geq 0$, **and** $\{g_i(x) - b_i\}_{i=1}^k \leq 0$
if the *ith* constraint is binding then,
$\lambda_i > 0$, $g_i(x) - b_i = 0$, and $\lambda_i(g_i(x) - b_i) = 0$. Otherwise,
$\lambda_i = 0$, $g_i(x) - b_i < 0$, and $\lambda_i(g_i(x) - b_i) = 0$.

2^k combinations

they are k binding ,$(k-1)$binding,$(k-2)$binding
,...,$(k-k+1)$binding,
$(k-k)$binding.

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Forms

- For a maximization problem; Max $f(x)$ s.t. $g(x) \leq 0$.
  $\mathscr{L}_{max} = f(x) + \lambda(-g(x))$ same as $\mathscr{L}_{max} = f(x) - \lambda(g(x))$
  Equivalently,
  $\mathscr{L}_{min} = -f(x) + \lambda(-g(x))$ or $\mathscr{L}_{min} = -f(x) - \lambda(g(x))$
- For a minimization problem; Min $f(x)$ s.t. $g(x) \leq 0$.
  $\mathscr{L}_{min} = f(x) + \lambda(-g(x))$ same as $\mathscr{L}_{min} = f(x) - \lambda(g(x))$
  Equivalently,
  $\mathscr{L}_{max} = -f(x) + \lambda(-g(x))$ or $\mathscr{L}_{max} = -f(x) - \lambda(g(x))$

Remarks (Personal Tricks):
1. The variances with equality constraints is due to the fact the inequality constraints are converted to equality constraints.
2. Generally and irrespective of Max. or Min., when you use $\{g_i(x)\}_{i=1}^{k} \leq 0$ then $sign(\lambda) < 0$ but if $\{g_i(x)\}_{i=1}^{k} \geq 0$ then $sign(\lambda) > 0$.
3. Equivalent cases corresponds to $-f(x)$ and not $-(\mathscr{L}_{max})$.

Inequaity Constrained Optimization Example

$$(x^\star, y^\star) \in \arg\max_{x,y} 3x + 4y$$

subject to $x^2 + y^2 \leq 4$, and $x \geq 1$
The CQs can be checked before or after obtaining the optimal solution.

Soln.

The constraints can be transformed to $x^2 + y^2 \leq 4$, and $-x \leq -1$

$\mathscr{L} = 3x + 4y - \lambda_1(x^2 + y^2 - 4) - \lambda_2(1 - x)$

FONCs

$\mathscr{L}_x = 3 - 2x\lambda_1 + \lambda_2 = 0$

$\mathscr{L}_y = 4 - 2y\lambda_1 = 0$

...

KKTCs

$\lambda_1 \geq 0$, $x^2+y^2-4 \leq 0$, and $\lambda_1(x^2+y^2-4)=0$

$\lambda_2 \geq 0$, $1-x \leq 0$, and $\lambda_2(1-x)=0$

2^k **combinations**

a.) $\lambda_1 > 0$ and $\lambda_2 > 0$

then we solve

$3-2x\lambda_1+\lambda_2=0$... eqn.(1)

$4-2y\lambda_1=0$ ... eqn.(2)

$x^2+y^2-4=0$ ... eqn.(3)

$1-x=0$ ... eqn.(4)

from eqn.(4) we get $x=1$. we use it in eqn.(3) to get $y=\sqrt{3}$. We use y in eqn.(2) to get $\lambda_1=\frac{2\sqrt{3}}{3}$. We use x and λ_1 in eqn.(1) to get $\lambda_2=\frac{4\sqrt{3}-9}{3}<0$. Recall we are solving under $\lambda_2>0$, so $\lambda_2=\frac{4\sqrt{3}-9}{3}$ does not satisfy this binding condition. Hence $(x,y,\lambda_1,\lambda_2)=(1,\sqrt{3},\frac{2\sqrt{3}}{3},\frac{4\sqrt{3}-9}{3})$ & $(1,-\sqrt{3},\frac{2\sqrt{3}}{3},\frac{4\sqrt{3}-9}{3})$ are not feasible solutions to the problem

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2^k **combinations**

b.) $\lambda_1 > 0$ and $\lambda_2 = 0$

then we solve

$3 - 2x\lambda_1 = 0$... eqn.(1)

$4 - 2y\lambda_1 = 0$ ... eqn.(2)

$x^2 + y^2 - 4 = 0$ ... eqn.(3)

$1 - x < 0$ ... exp.(4)

from eqn.(1) we get $\lambda_1 = \frac{3}{2x}$ and from eqn.(2) we get $\lambda_1 = \frac{2}{y}$. Equating these, we get $x = \frac{3y}{4}$. Also, from $eqn.(3)$, $x = \pm\sqrt{4 - y^2}$. Equating these two gives $y = \pm\frac{8}{5}$. Then $\lambda_1 = \frac{5}{4}$ and $x = \frac{6}{5}$. Hence $(\lambda_1, \lambda_2, x, y) = (\frac{5}{4}, 0, \frac{6}{5}, \frac{8}{5})$ is a feasible solutions to the problem since $x > 1$ from exp.(4).

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2^k combinations

c.) $\lambda_1 = 0$ and $\lambda_2 > 0$

then we solve

$3 - \lambda_2 = 0$... eqn.(1)

$4 = 0$ ... eqn.(2)

$x^2 + y^2 - 4 < 0$ ... exp.(3)

$1 - x = 0$ ... eqn.(4)

from eqn.(4) and eqn(1) we get $x = 1$ and $\lambda_2 = 3$. From eqn.(3) we get $y < \pm\sqrt{3}$ and from eqn.(2), we see $4 = 0$. But this is not true as $4 \neq 0$. Hence, no solution exists for this case.

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2^k combinations

d.) $\lambda_1 = \lambda_2 = 0$

then we solve

$3 = 0$... eqn.(1)

$4 = 0$ ... eqn.(2)

$x^2 + y^2 - 4 < 0$ ... exp.(3)

$1 - x < 0$ ... eqn.(4)

from eqn.(1) and eqn.(2), no solution exists for this case.

checking for the maximizer

In all, the only feasible solution set (and unique maximizer) we have is $(\lambda_1, \lambda_2, x, y) = (\frac{5}{4}, 0, \frac{6}{5}, \frac{8}{5})$, Therefore, the value function becomes $f(x = \frac{6}{5}, y = \frac{8}{5}) = 10$. we can also rewrite the lagrangian function as $\mathscr{L} = 3x + 4y - \frac{5}{4}x^2 - \frac{5}{4}y^2 + 5$. Finally we can show that the lagrange function is concave in x and y using appropriate methodologies.

Constraint Qualification Tests

1. $k = 2$ **constraints are binding**
i.e. $x^2 + y^2 = 4$, and $1 - x = 0$
$J_{1,2} = Dh_{E(x^\star, y^\star)} = \begin{bmatrix} 2x & 2y \\ -1 & 0 \end{bmatrix}_{x=x^\star, y=y^\star}$ or $J_1 = \begin{bmatrix} 2x & 2y \end{bmatrix}$,
$J_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}$, are full rank hence, CQ1 is satisfied
2. $k = 1$ **constraints are binding**
2.1. $x^2 + y^2 = 4$ **is binding**, $J_1 = \begin{bmatrix} 2x & 2y \end{bmatrix}$ full rank
2.2. $1 - x = 0$ **is binding**, $J_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}$, full rank
3. $k = 0$ **is binding**
No jacobian matrix. **Therefore, CQ1 is satisfied**
In addition, is **CQ2 satisfied**?

Other Examples

$$\arg\max_{x_1,x_2} \sqrt{x_1 x_2}$$

subject to $x_1^2 + x_2^2 \leq 5$, and $x_1, x_2 \geq 0$

$$\arg\min_{x_1,x_2} 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2$$

subject to $x_1^2 + x_2^2 \leq 5$, and $3x_1 + x_2 \leq 6$

$$\arg\max_{x,y} 2x^2 + 3xy$$

subject to $\frac{1}{2}x^2 + y \leq 4$, and $-y \leq -2$

$$\arg\max_{x_1,x_2} x_1^2 + 2x_2 + 2x_3^2$$

subject to $2x_1^2 - x_2^2 - 3x_3 = 0$, and $x_2 - x_3 = 3$

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λ is the shadow price on the constraint. It is the ratio of the change in the value function to the change in the constraint, $\lambda = \frac{dV}{dc} = \frac{df(x^\star)}{dc}$.

Recall this problem

$$\arg\max_{c_1 \geq 0, c_2 \geq 0} U(c_1, c_2)$$

$$\text{subject to } p_1c_1 + p_2c_2 \leq I,\ c_1 > 0\ ,\ \text{and } c_2 > 0$$

$$\mathscr{L}_{max} = U(c_1, c_2) + \lambda(I - p_1c_1 - p_2c_2) \ldots eqn(1)$$

FONCs

$$\mathscr{L}_{c_1} = U_{c_1}(c_1, c_2) - p_1\lambda = 0 \ldots eqn.(2)$$
$$\mathscr{L}_{c_2} = U_{c_2}(c_1, c_2) - p_2\lambda = 0 \ldots eqn.(3)$$
$$\mathscr{L}_{\lambda} = I - p_1c_1 - p_2c_2 = 0 \ldots eqn.(4)$$

Combining these three equations eqn.(2), eqn.(3), & eqn.(4) would produce the maximizers $c_1^\star(p_1, p_2, l)$, $c_2^\star(p_1, p_2, l)$, and $\lambda^\star(p_1, p_2, l)$.

The value function,

$$V(p_1, p_2, l) = \underset{c_1>0, c_2>0}{\arg\max}\, U(c_1, c_2)$$

becomes

$V(p_1, p_2, l) = U(c_1^\star(p_1, p_2, l), c_2^\star(p_1, p_2, l)) + \lambda^\star(p_1, p_2, l)\{I - p_1 c_1^\star(p_1, p_2, l)$
$-p_2 c_2^\star(p_1, p_2, l)\}$

Our interest now is to see how the value function would change when the income constraint changes i.e. $\frac{\delta V(p_1, p_2, l)}{\delta I}$.

$\frac{\delta V(p_1, p_2, l)}{\delta I} = U_{c_1^\star} \frac{\delta c_1^\star}{\delta I} + U_{c_2^\star} \frac{\delta c_2^\star}{\delta I} + \lambda^\star (1 - p_1 \frac{\delta c_1^\star}{\delta I} - p_2 \frac{\delta c_2^\star}{\delta I}) +$
$\frac{\delta \lambda^\star}{\delta I}(I - p_1 c_1^\star - p_2 c_2^\star)$

Recall, $I - p_1 c_1^\star - p_2 c_2^\star = 0$, then we can rewrite the equation as

$\frac{\delta V(p_1, p_2, l)}{\delta I} = U_{c_1^\star} \frac{\delta c_1^\star}{\delta I} + U_{c_2^\star} \frac{\delta c_2^\star}{\delta I} + \lambda^\star (1 - p_1 \frac{\delta c_1^\star}{\delta I} - p_2 \frac{\delta c_2^\star}{\delta I})$

$\frac{\delta V(p_1, p_2, l)}{\delta I} = U_{c_1^\star} \frac{\delta c_1^\star}{\delta I} + U_{c_2^\star} \frac{\delta c_2^\star}{\delta I} + \lambda^\star - p_1 \lambda^\star \frac{\delta c_1^\star}{\delta I} - p_2 \lambda^\star \frac{\delta c_2^\star}{\delta I}$

$\frac{\delta V(p_1, p_2, l)}{\delta I} = (U_{c_1^\star} - p_1 \lambda^\star) \frac{\delta c_1^\star}{\delta I} + (U_{c_2^\star} - p_2 \lambda^\star) \frac{\delta c_2^\star}{\delta I} + \lambda^\star.$

Since $c_1^\star(p_1, p_2, l)$, $c_2^\star(p_1, p_2, l)$, and $\lambda^\star(p_1, p_2, l)$ are the values that satisfy eqn.(2), eqn.(3), & eqn.(4), then $\frac{\delta V(p_1, p_2, l)}{\delta I} = \lambda^\star$

Examples

Solving

$$\arg\max_{x} f(x) = x^2$$

s.t $c \geq x$ and $x > 0$
gives that $x^\star = c$ and $\lambda^\star = 2x = 2c$.
However, the value function $V(c) = c^2$ then $\frac{dV(c)}{dc} = 2c = \lambda^\star$

Pseudo Midter is on 20/09/2019
GoodLuck!!!