Convexity, Concavity and Equality Optimization

Introductory Mathematical Economics

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Outline

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Definition: Convex Set

A set S $\subset \mathbb{R}^n$ is convex if $z \epsilon$ S $\forall$ $x \epsilon$ S and $x\prime \epsilon$ S. Where x, $x\prime \epsilon \mathbb{R}^n$, $\alpha \epsilon$ [0,1] and convex combination $z = \alpha x + (1 - \alpha)x\prime$.

Remarks:

1). Get arbitrary two (2) points. Eg in
$\mathbb{R}^1$: x_1 and x_2,
$\mathbb{R}^2$: (x_1, y_1) and (x_2, y_2),
$\mathbb{R}^3$: (x_1, y_1, z_1) and (x_2, y_2, z_2),
etc.
2). Get the convex combination(s). Eg in
$\mathbb{R}^1$: $Z = \alpha x_1 + (1 - \alpha)x_2$,
$\mathbb{R}^2$: $Z = (\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_2)$,
$\mathbb{R}^3$: $Z = (\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_2, \alpha z_1 + (1 - \alpha)z_2)$,
etc.

Remarks Cont.

3). Use the definition on the arbitrary points i.e. what you know. Eg if $\{(x, y)|x \geq 0, f(x) \geq y\}$, then, $x_1 \geq 0, f(x_1) \geq y_1$ $x_2 \geq 0$ and $f(x_2) \geq y_2$.

...

4). Use the definitions of the curvature and what you know (i.e. point 3) to show that the convex combination, Z, belongs to the set; it satisfies $z_1 \geq 0$ and $f(z_1) \geq z_2$, where
$Z = (z_1, z_2) = (\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_2)$

...

5). Use the curvature definitions to show the function is concave (convex) i.e. $f(Z) \;\geq\; (\leq)\; \alpha f(x_1) + (1 - \alpha)f(x_2)$ holds for concavity (convexity), where $Z = \alpha x_1 + (1 - \alpha)x_2$.

Examples

Show whether these sets are convex sets.

- $S = [10, 30]$, $S = (10, 30)$ and $S = (10, 30]$
- $Q = \mathbb{R}$ and $P = \mathbb{N}$
- X ϵ $\mathbb{R}^2$, $X = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 4\}$ and
  Y ϵ $\mathbb{R}^2$, $Y = \{(y_1, y_2) | y_1^2 + y_2^2 \geq 4\}$
- Set of feasible consumption bundles $\Gamma(p, l) = \{c | c_i \geq 0, pc \leq l\}$, where $c, p \ \epsilon \ \mathbb{R}^n_+$ and $l \ \epsilon \ \mathbb{R}_+$

Remarks:

1.Feasible Choice set is often convex.

2.Sketching the set is advisable.

Outline

Definition: Convex Function

For a convex domain D, a function f is convex over D if $f(z) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$ $\forall$ α ϵ [0,1], x_1, x_2 ϵ D and $z = \alpha x_1 + (1-\alpha)x_2$ is the convex combination.

Examples

Show whether these functions are convex functions

- $f(x) = x + 2$
- $k(x) = 2 + x^2$ and x ϵ $\mathbb{R}^1$
- $h(x) = sin(x)$ and x ϵ $(0, 2\pi)$
- $g(x) = sin(x)$ and x ϵ $(\pi, 2\pi)$
- Show that $f(x) = 1 - x^2$ is a concave function

Remarks:

1.Sketching the function over its domain is advisable.

Outline

Unconstrained Curvature: One Variable

- Concavity is defined as $f\prime \geq / \leq 0$ and $f\prime\prime \leq 0$ over its domain.
- Convexity is defined as $f\prime \geq / \leq 0$ and $f\prime\prime \geq 0$ over its domain.
- We have strict concavity/convexity if $f\prime\prime < 0/f\prime\prime > 0$ respectively.

Unconstrained Curvature: Two Variables

- Concavity is defined as $f_{x_1,x_1} \leq 0$, $f_{x_2,x_2} \leq 0$ and $f_{x_1,x_1} f_{x_2,x_2} - (f_{x_1,x_2})^2 \geq 0$ over its domain.
- Convexity is defined as $f_{x_1,x_1} \geq 0$, $f_{x_2,x_2} \geq 0$ and $f_{x_1,x_1} f_{x_2,x_2} - (f_{x_1,x_2})^2 \geq 0$ over its domain.
- We have strict concavity $f_{x_1,x_1} < 0$, $f_{x_2,x_2} < 0$ and $f_{x_1,x_1} f_{x_2,x_2} - (f_{x_1,x_2})^2 > 0$ over its domain.
- We have strict convexity $f_{x_1,x_1} > 0$, $f_{x_2,x_2} > 0$ and $f_{x_1,x_1} f_{x_2,x_2} - (f_{x_1,x_2})^2 > 0$ over its domain.

Unconstrained Curvature: More than two Variable

We adopt the Hessian matrix, H, of f at $\mathbf{x}$

$$H = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_1 \delta x_n} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} & \cdots & \frac{\delta^2 f}{\delta x_2 \delta x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta^2 f}{\delta x_n \delta x_1} & \frac{\delta^2 f}{\delta x_n \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_n^2} \end{bmatrix}$$

- Concavity is defined as negative semi-definite i.e. $(-1)^k \Delta_k \geq 0 \ \forall$ k
- Convexity is defined as positive semi-definite i.e. $\Delta_k \geq 0 \ \forall$ k
- Strict concavity is defined as negative definite i.e. $(-1)^k D_k > 0 \ \forall$ k
- Strict convexity is defined as positive definite i.e. $D_k > 0 \ \forall$ k.

Where D_k is the **leading principal minor** and Δ_k is the **arbitrary principal minor** of order k. $k = 1, 2, \ldots, n$. n is the dimension of the matrix.

Constrained Curvature

We adopt the Bordered Hessian matrix $\mathbf{H}$ of $\mathscr{L}$ at $\mathbf{x}$

$$\mathbf{H} = \begin{bmatrix} \mathbf{D}_x^2 \mathscr{L}(x^*, \lambda^*) & \mathbf{D}g(x^*) \\ (\mathbf{D}g(x^*))^T & \mathbf{0} \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc}
\mathscr{L}_{x_1,x_1} & \mathscr{L}_{x_1,x_2} & \cdots & \mathscr{L}_{x_1,x_n} & \mathscr{L}_{x_1,\lambda_1} & \mathscr{L}_{x_1,\lambda_2} & \cdots & \mathscr{L}_{x_1,\lambda_k} \\
\mathscr{L}_{x_2,x_1} & \mathscr{L}_{x_2,x_2} & \cdots & \mathscr{L}_{x_2,x_n} & \mathscr{L}_{x_2,\lambda_1} & \mathscr{L}_{x_2,\lambda_2} & \cdots & \mathscr{L}_{x_2,\lambda_k} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathscr{L}_{x_n,x_1} & \mathscr{L}_{x_n,x_2} & \cdots & \mathscr{L}_{x_n,x_n} & \mathscr{L}_{x_n,\lambda_1} & \mathscr{L}_{x_n,\lambda_2} & \cdots & \mathscr{L}_{x_n,\lambda_k} \\
-- & -- & -- & -- & -- & -- & -- & -- \\
\mathscr{L}_{\lambda_1,x_1} & \mathscr{L}_{\lambda_1,x_2} & \cdots & \mathscr{L}_{\lambda_1,x_n} & \mathscr{L}_{\lambda_1,\lambda_1} & \mathscr{L}_{\lambda_1,\lambda_2} & \cdots & \mathscr{L}_{\lambda_1,\lambda_k} \\
\mathscr{L}_{\lambda_2,x_1} & \mathscr{L}_{\lambda_2,x_2} & \cdots & \mathscr{L}_{\lambda_2,x_n} & \mathscr{L}_{\lambda_2,\lambda_1} & \mathscr{L}_{\lambda_2,\lambda_2} & \cdots & \mathscr{L}_{\lambda_2,\lambda_k} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathscr{L}_{\lambda_k,x_1} & \mathscr{L}_{\lambda_k,x_2} & \cdots & \mathscr{L}_{\lambda_k,x_n} & \mathscr{L}_{\lambda_k,\lambda_1} & \mathscr{L}_{\lambda_k,\lambda_2} & \cdots & \mathscr{L}_{\lambda_k,\lambda_k}
\end{array}\right]$$

...

- Concavity is defined as negative semi-definite i.e. if $sign(det(\mathbf{H})) = (-1)^n$ and $sign$(last $(n-k)$ leading principal minors of $\mathbf{H}$) alternates.
- Convexity is defined as positive semi-definite i.e. if $sign$(last $(n-k)$ leading principal minors of $\mathbf{H}$) are same as $(-1)^k$

Remarks:

1. Where n is the # of choice variables and k is the # of constraints.
2. FONC corresponds to critical points while SOSC corresponds to the curvature at $\mathbf{x}^*$.

Examples

Determine the curvature of the following

- $f(x) = 3x^3 - 2x^2 + 8$
- $f(x, y) = 2x - y - x^2 + 2xy - y^2$
- $Q(x, y, z) = -x^2 + 6xy + 8yz - 9y^2 - 2z^2$
- See the note on review of Linear Algebra (matrices & determinants)

Outline

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Setting up the Lagrange Method

Remarks (Personal Tricks):
1. Otherwise defined, ensure that $sign(\lambda) \equiv sign(\alpha)$, where α is a constant.
...
2. Preferably; for maximization problems let $sign(\lambda) \equiv sign(\alpha) > 0$ and $sign(\lambda) \equiv sign(\alpha) < 0$ for minimization problems.
...
3. Equivalent scenarios; let $g(x) = -f(x)$ then $\max_x f(x)$ is same as $\min_x g(x)$ and vice versa.

Examples

- For a maximization problem; $\mathscr{L}_{max} = f(\mathbf{x}) + \lambda(I - \mathbf{px})$.
  Equivalently,
  $\mathscr{L}_{min} = -(\mathscr{L}_{max}) = -f(\mathbf{x}) - \lambda(I - \mathbf{px})$ or
  $\mathscr{L}_{min} = -(\mathscr{L}_{max}) = -f(\mathbf{x}) + \lambda(\mathbf{px} - I)$
- For a minimization problem; $\mathscr{L}_{min} = wl + rk - \lambda(f(k,l) - Q)$.
  Equivalently,
  $\mathscr{L}_{max} = -(\mathscr{L}_{min}) = -wl - rk + \lambda(f(k,l) - Q)$ or
  $\mathscr{L}_{max} = -(\mathscr{L}_{min}) = -wl - rk - \lambda(Q - f(k,l))$

Examples

Show the FONC and SOSC of the following optimization problems:

- $$(c_1^*, c_2^*) \,\epsilon \underset{c_1 \geq 0, c_2 \geq 0}{\arg\max}\, U(c_1, c_2)$$

  subject to $p_1 c_1 + p_2 c_2 = I$.

- $$(x_1^*, x_2^*) \,\epsilon \underset{x_1 \geq 0, x_2 \geq 0}{\arg\min}\;\; w_1 x_1 + w_2 x_2$$

  subject to $f(x_1, x_2) = y$