Introduction to Mathematical Economics
Review on Linear Algebra (Matrices and Determinants)

TA Session

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Matrix Algebra: Addition and subtraction

Addition, subtraction of matrices:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \dots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & \dots & a_{1n} \pm b_{1n} \\ \vdots & a_{ij} \pm b_{ij} & \vdots \\ a_{k1} \pm b_{k1} & \dots & a_{kn} \pm b_{kn} \end{pmatrix}$$

Matrix Algebra: Scalar Multiplication

Scalar Multiplication:

$$r\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \dots & ra_{1n} \\ \vdots & ra_{ij} & \vdots \\ ra_{k1} & \dots & ra_{kn} \end{pmatrix}$$

Matrix Algebra: Matrix Multiplication

Matrix Multiplication:

- The matrix product AB is well defined if and only if:

$$\text{number of columns of } A = \text{number of rows of } B$$

- Let A be a $k \times m$ matrix and B a $m \times n$ matrix. Then AB is a $k \times n$ matrix and its (i, j)th entry is

$$\begin{pmatrix} a_{i1} & \dots & a_{im} \end{pmatrix} \cdot \begin{pmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{pmatrix} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj}$$

Matrix Algebra: Laws of Matrix Algebra

Laws of Matrix Algebra:

- **Associative Laws:** $(A + B) + C = A + (B + C)$; $(AB)C = A(BC)$.
- **Commutative Law for Addition:** $A + B = B + A$, but generally $\mathbf{AB} \neq \mathbf{BA}$.
- **Distributive Laws:** $(A + B)C = AC + BC$; $A(B + C) = AB + AC$.

Matrix Algebra: Transpose

Transpose

- The **transpose** of a $k \times n$ matrix A: $A^\top$. It is a $n \times k$ matrix obtained by interchanging the rows and columns of A.
- $(A \pm B)^\top = A^\top \pm B^\top$;
- $(A^\top)^\top = A$;
- $(rA)^\top = rA^\top$;
- $(AB)^\top = B^\top A^\top$.

Theorem 8.1

$$(AB)^\top = B^\top A^\top$$

Special Kinds of Matrices

Special matrices

- square matrix; column matrix; row matrix;
- diagonal matrix; upper-triangular matrix; lower-triangular matrix
- symmetric matrix; idempotent matrix; permutation matrix; nonsingular matrix.

Systems of Equations in Matrix Form

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \Longrightarrow A\mathbf{x} = \mathbf{b}$$

Inverse of Square Matrix

Definition:

- Let A be an $n \times n$ matrix. The $n \times n$ matrix B is an **inverse** for A if $AB = BA = I$.
- Let A be an $k \times n$ matrix. The $n \times k$ matrix B is a **right inverse** for A if $AB = I$.
- Let A be an $k \times n$ matrix. The $n \times k$ matrix c is a **left inverse** for A if $CA = I$.

Inverse of Square Matrix

Theorem 8.5

An $n \times n$ matrix A can have at most one inverse.

Theorem 8.6

If an $n \times n$ matrix A is invertible, then it is nonsingular, and the unique solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Theorem 8.7

If an $n \times n$ matrix A is non-singular, then it is invertible.

- Example 8.3 and 8.4
- Exercise 8.19 and 8.28

Inverse of Square Matrix

Theorem 8.9

For any square matrix A, the following statements are equivalent:

- (a) A is invertible.
- (b) A has a right inverse.
- (c) A has a left inverse.
- (b) Every system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every b.
- (e) Every system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every b.
- (f) A is nonsingular.
- (g) A has a maximal rank n.

Inverse of Square Matrix

Theorem 8.10

Let A and B are square invertible matrices. Then,

- (a) $(A^{-1})^{-1} = A$.
- (b) $(A^{\top})^{-1} = (A^{-1})^{\top}$.
- (c) AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Inverse of Square Matrix

Theorem 8.11

If A is invertible:

- (a) A^m is invertible for any integer m and

$$(A^m)^{-1} = (A^{-1})^m = A^{-m}$$

- (b) for any integers r and s,

$$A^r A^s = A^{r+s}$$

- (c) for any scaler $r \neq 0$, rA is invertible and

$$(rA)^{-1} = \frac{1}{r} A^{-1}$$

Determinant

Defining the Determinant

- Let A be an $n \times n$ matrix. Let A_{ij} be an $(n-1) \times (n-1)$ submatrix obtained by deleting i-th row and j-th column from A. Then,
  - the scalar $M_{ij} \equiv detA_{ij}$ is called the (i,j)th **minor** of A,
  - the scaler $C_{ij} \equiv (-1)^{i+j} detA_{ij}$ is called the (i,j)th **cofactor** of A.
- The **determinant** of an $n \times n$ matrix A is given by

$$detA = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Computing the Determinant

Theorem 9.3

A square matrix is nonsingular if and only if its determinant is nonzero.

Uses of the Determinant

- The $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (i, j)th cofactor of A, is called the **adjoint** of A and is written **adj** A.

Theorem 9.4

Let A be a nonsingular matrix. Then,

- (a) $A^{-1} = \frac{1}{\det A} \cdot \text{adj} A$, and
- (b) **(Cramer's rule)** the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det B_i}{\det A}, \qquad \text{for } i = 1, \cdots, n,$$

where B_i is the matrix A with the RHS $\mathbf{b}$ replacing the i-th column of A.

- Example 9.3 and 9.4.

Uses of the Determinant

Theorem 9.5

Let A be a square matrix. Then,

- (a) $\det A^\top = \det A$,
- (b) $\det(AB) = (\det A)(\det B)$, and
- $\det(A + B) \neq \det A + \det B$, in general.

- IS-LM analysis via Cramer's rule.
- Exercise 9.11.

Leading Principal Minor

A square matrix, $\{A\}_{ij}$ has n leading principal minors. Where $n = i = j$

Given that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The leading principal minors are:

$$D_1 = \begin{bmatrix} a_{11} \end{bmatrix}, D_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} and\, D_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Arbitrary Principal Minor

A square matrix, $\{A\}_{ij}$ has $k - order$ arbitrary principal minors. Where $k = 1, 2, ..., n$ and $n = i = j$. This is derived from cancelling different and unique equal $(n - k)$ number of rows and columns. Using the already defined $\{A\}_{ij}$.

The arbitrary principal minors are:

$$\Delta_1^1 = \begin{bmatrix} a_{11} \end{bmatrix}, \Delta_1^2 = \begin{bmatrix} a_{22} \end{bmatrix}, and\, \Delta_1^1 = \begin{bmatrix} a_{33} \end{bmatrix}$$

$$\Delta_2^1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \Delta_2^2 = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, and\, \Delta_2^3 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

and

$$\Delta_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$