# Stochastic Dynamic Optimization:

## Recursive Value Function

Introductory Mathematical Economics

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# Outline

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Definition of the Fundamental Recursive Relation:

**For each time t**

$$V_t(s_t) = \underset{a_t}{\operatorname{argmax}}[U(s_t, a_t)^* + \delta^\dagger EV_{t\pm^\ddagger 1}(s_{t+1})]$$

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* This is the reward (utility, profit, welfare, et detera) function

† This is the discount factor, not the discount rate

‡ This depends on the initial condition provided

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## Model Setup

- $U(s_t, a_t) = pa_t - \frac{ca_t^2}{s_t}$
- $s_{t+1} = s_t - a_t$
- $V_0 = 0$
- $p = 1$, $c = 10$, $\delta = 1$

Where $s_t$ is the exhaustible resource stock. $a_t$ is the extraction

Find the optimal policies

$$a_t = \pi(s_t)$$

for $t = 1, 2, 3, ..., T$

# Solution

$$V_t(s_t) = \underset{a_t}{\text{argmax}}[U(s_t, a_t) + \delta EV_{t-1}(s_{t+1})]$$

$$V_1(s_1) = \underset{a_1}{\text{argmax}}[pa_1 - \frac{ca_1^2}{s_1} + \delta EV_0(s_2)]$$

recall $p = 1$, $c = 10$, $\delta = 1$, and $V_0(s_2) = 0$

$$V_1(s_1) = \underset{a_1}{\text{argmax}}[a_1 - \frac{10a_1^2}{s_1} + 1 * 0]$$

FONC (w.r.t choice variable):

$$1 - \frac{20a_1}{s_1} = 0$$

$$a_1 = \pi_1(s_1) = \frac{1}{20}s_1$$

$$V_1(s_1) = \frac{s_1}{20} - \frac{10(\frac{s_1}{20})^2}{s_1} = \frac{s_1}{40}$$

$$V_2(s_2) = \underset{a_2}{\text{argmax}}[U(s_2, a_2) + \delta EV_1(s_3)]$$

$$V_2(s_2) = \underset{a_2}{\text{argmax}}[pa_2 - \frac{ca_2^2}{s_2} + \delta EV_1(s_3)]$$

recall $p = 1$, $c = 10$, $\delta = 1$, $s_{t+1} = s_t - a_t$

$$V_2(s_2) = \underset{a_2}{\text{argmax}}[a_2 - \frac{10a_2^2}{s_2} + 1 * V_1(s_2 - a_2)]$$

recall $V_1(s_1) = \frac{s_1}{40} \rightarrow V_1 = \frac{1}{40}$

$$V_2(s_2) = \underset{a_2}{\text{argmax}}[a_2 - \frac{10a_2^2}{s_2} + 1 * \frac{1}{40}(s_2 - a_2)]$$

FONC (w.r.t choice variable):

$$1 - \frac{20a_2}{s_2} - \frac{1}{40} = 0$$

$$a_2 = \pi_2(s_2) = \frac{39}{800}s_2$$

$$V_2(s_2) = \frac{39s_2}{800} - \frac{10(\frac{39s_2}{800})^2}{s_2} + 1 * \frac{1}{40}(s_2 - \frac{39s_2}{800}) = \frac{3121}{64000}s_2$$

$$V_3(s_3) = \underset{a_3}{\text{argmax}}[U(s_3, a_3) + \delta E V_2(s_4)]$$

$$V_3(s_3) = \underset{a_3}{\text{argmax}}[pa_3 - \frac{ca_3^2}{s_3} + \delta E V_2(s_4)]$$

recall $p = 1$, $c = 10$, $\delta = 1$, $s_{t+1} = s_t - a_t$

$$V_3(s_3) = \underset{a_3}{\text{argmax}}[a_3 - \frac{10a_3^2}{s_3} + 1 * V_2(s_3 - a_3)]$$

recall $V_2(s_2) = \frac{3121}{64000} s_2 \to V_1 = \frac{3121}{64000}$

$$V_3(s_3) = \underset{a_3}{\text{argmax}}[a_3 - \frac{10a_3^2}{s_3} + 1 * \frac{3121}{64000}(s_3 - a_3)]$$

FONC (w.r.t choice variable):

$$1 - \frac{20a_3}{s_3} - \frac{3121}{64000} = 0$$

$$a_3 = \pi_3(s_3) = \frac{60,879}{1,280,000} s_2$$

$$V_3(s_3) = \frac{60,879 s_2}{1,280,000} - \frac{10(\frac{60,879 s_2}{1,280,000})^2}{s_2} + 1 * \frac{3121}{64000}(s_3 - \frac{60,879 s_2}{1,280,000}) = 0.07142 s_3$$

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## Model Setup

- $U(y_t, c_t)$ is the social welfare
- $x_t = y_t - c_t$ is investment
- $y_{t+1} = f(y_t - c_t)$ stock transition equation
- $\delta = \frac{1}{1+r}$ discount factor,(r = discount rate)

Where $y_t$ is the fish stock. $c_t$ is the harvest

Find the optimal harvest to maximize discounted sum of social welfare over time.

# Solution

$$V_t(y_t) = \underset{c_t, y_t}{\text{argmax}}[U(c_t, y_t) + \delta E V_{t+1}(y_{t+1})]$$

recall $y_{t+1} = f(x_t) = f(y_t - c_t)$

$$V_t(y_t) = \underset{c_t, y_t}{\text{argmax}}[U(c_t, y_t) + \delta E V_{t+1}(y_{t+1})]$$

$$V_t(y_t) = \underset{c_t, y_t}{\text{argmax}}[U(c_t, y_t) + \delta E V_{t+1}(f(y_t - c_t))]$$

FONC (w.r.t choice variable)

$$c_t : U_{c_t}(c_t, y_t) + \delta V_{y_{t+1}}(f(y_t - c_t)) f_{x_t}(y_t - c_t)(-1) = 0 \quad (1)$$

$$U_{c_t}(c_t, y_t) = \delta V_{y_{t+1}}(f(y_t - c_t)) f_{x_t}(y_t - c_t) \quad (1^\star)$$

By Envelope Theorem i.e. differentiate both sides w.r.t the state variable

$$y_t : V_{y_t}(y_t) = U_{y_t}(c_t, y_t) + \delta V_{y_{t+1}}(f(y_t - c_t)) f_{x_t}(y_t - c_t)(1) \quad (2)$$

Example 2 (combining state variable constraints)

combine (1) and (2)

$$V_{y_t}(y_t) = U_{y_t}(c_t, y_t) + U_{c_t}(c_t, y_t) \quad (3)$$

rewrite the optimality conditions of $c_t$ using (3)

$$U_{c_t}(c_t, y_t) = \delta[U_{y_{t+1}}(c_{t+1}, y_{t+1}) + U_{c_{t+1}}(c_{t+1}, y_{t+1})]f_{x_t}(y_t - c_t) \quad (4)$$

At Steady Sate, from (4)

$$\frac{1}{\delta} = [1 + \frac{U_y(c, y)}{U_c(c, y)}]f_x(y - c)$$

$$1 + r = [1 + \frac{U_y(c, y)}{U_c(c, y)}]f_x(y - c)^{\S}$$

by assuming $U_{y_t}(c_t, y_t) = 0$, the steady state condition reduces to

$$1 + r = f_x(y - c)$$

We could also show the steady state condition (a bit) differently while assuming $U_{y_t}(c_t, y_t) = 0$ in (3) and use $(1^{\star})$

$$V_{y_t}(y_t) = U_{c_t}(c_t, y_t) = \delta V_{y_{t+1}}(f(y_t - c_t))f_{x_t}(y_t - c_t) \quad (5)$$

§This is therefore, the steady state condition for this problem

Example 2 (combining state variable constraints)

$$V_{y_t}(y_t) = \frac{1}{1+r}[V_{y_{t+1}}(f(y_t - c_t))f_{x_t}(y_t - c_t)]$$

$$1 + r = \frac{V_{y_{t+1}}(f(y_t - c_t))f_{x_t}(y_t - c_t)}{V_{y_t}(y_t)} \quad (6)$$

$$1 + r = \frac{V_{y_{t+1}}(f(y_t - c_t))}{V_{y_t}(y_t)} f_{x_t}(y_t - c_t)$$

$$1 + r = \frac{V_{y_{t+1}}(f(y_t - c_t)) + V_{y_t}(y_t) - V_{y_t}(y_t)}{V_{y_t}(y_t)} f_{x_t}(y_t - c_t)$$

let $\Delta V_{y_t} = V_{y_{t+1}}(y_{t+1}) - V_{y_t}(y_t)$ then,

$$1 + r = [1 + \frac{\Delta V_{y_t}(y_t)}{V_{y_t}(y_t)}] f_{x_t}(y_t - c_t)$$

At Steady State, $\Delta V_{y_t}(y_t) = 0$ and this reduces to

$$1 + r = f_x(y - c)^{\P}$$

¶This is the same steady state. $\delta = \frac{1}{1+r}$

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## Model Setup

- $U(q_t, c_t) = \int_0^{c_t} -P(x)dx - K(s_t)$ is the social welfare
- $q_t = c_t + s_t$ is the seed crop size
- $s_t = (1+g)s_{t-1} - c_t$ stock transition equation
- $g = 6.5$

Where
$s_t$ : size of seed crop, $q_t$ : output, $c_t$ : consumption, $g$ : productivity, $P$ : price, $K(s_t)$ : cost of planting and storage and $r$ : interest rate.

Find the optimal action variable and state variable(s)

# Soution

The action variable is $c_t$ while the state variables are $s_t$ and $q_t$. The state variables are related through the action/choice variable [$q_t = c_t - s_t$]. Hence, solving for the optimal choice/action variable and one state variable could produce the optimal value of the other state variable. Let' solve for $c_t$ and $q_t$.

$$\underset{c_t, q_t}{\operatorname{argmax}} \int_0^{c_t} -P(x)d(x) - K(s_t)$$

s.t.

$$q_t = c_t + s_t$$

$$s_t = (1+g)s_{t-1} - c_t$$

Since we want to solve for $c_t$ and $q_t$, we could re-write the model setup as follows:

$$\underset{c_t,q_t}{\text{argmax}} \int_0^{c_t} -P(x)d(x) - K(q_t - c_t)$$

s.t.

$$q_t = (1+g)(q_{t-1} - c_{t-1})$$

The fundamental recursive form of this problem is

$$V(q_t) = \underset{c_t,q_t}{\text{argmax}}[U(c_t, q_t) + \delta EV_{t+1}(q_{t+1})]$$

$$V(q_t) = max_{q_t,c_t}[\int_0^{c_t} -P(x)dx - K(q_t - c_t) + \delta EV_{t+1}((1+g)(q_t - c_t)]$$

FONC (w.r.t choice variable, Leibnitz Rule)

$$-P(c_t) - K_{s_t}(q_t - c_t) \times -1 + \delta EV_{t+1 q_{t+1}}((1+g)(q_t - c_t))[1+g](-1) = 0$$

$$P(c_t) = K_{s_t}(q_t - c_t) - \delta EV_{t+1 q_{t+1}}((1+g)(q_t - c_t))[1+g] \quad (1)$$

Envelope Theorem(w.r.t the state variable)

$$V_{q_t}(q_t) = -K_{s_t}(q_t - c_t) + \delta E V_{t+1 q_{t+1}}((1+g)(q_t - c_t))[1+g] \quad (2)$$

from (1)

$$\delta E V_{t+1 q_{t+1}}((1+g)(q_t - c_t))[1+g] = K_{s_t}(q_t - c_t) - P(c_t) \quad (3)$$

Use (3) and (2)

$$V_{q_t}(q_t) = -K_{s_t}(q_t - c_t) + K_{s_t}(q_t - c_t) - P(c_t)$$

$$V_{q_t}(q_t) = -P(c_t) \quad (4)$$

Re-write (1) using (4)

$$P(c_t) = K_{s_t}(q_t - c_t) - \delta[-(P(c_{t+1}))][1+g]$$

$$P(c_t) = K_{s_t}(q_t - c_t) + \delta(P(c_{t+1}))[1+g]$$

$$P(c_t) - \delta P(c_{t+1}) = K_{s_t}(q_t - c_t) + g\delta P(c_{t+1})$$

$$\delta P(c_{t+1}) - P(c_t) = -g\delta P(c_{t+1}) - K_{s_t}(c_t - q_t) \quad (5)$$

from the original transition equations of $q_t$

$$q_{t+1} - q_t = gq_t - (1+g)c_t \quad (6)$$

Equations (5) and (6) are the Difference Equations (DCE) of the model. We could proceed further to plot the phase diagram and determine the steady state(s).

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What are the steady state conditions from (5) and (6) ?

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## Model Setup

$$\underset{f_t, a_t}{\operatorname{argmax}} \sum_{t=0}^{\infty} \delta^t [U(f_t) - C(a_t)]$$

s.t.

$$r_{t+1} = r_t - a_t$$

$$k_{t+1} = k_t + \gamma a_t - \beta f_t$$

$$0 \leq f_t \leq k_t$$

$$a_t \geq 0$$

Answer the following questions

# Solution

## a) Write the fundamental recursive form

The action variables are $a_t$ and $f_t$ while the state variables are $r_t$ and $k_t$. Therefore

$$V(r_t, k_t) = U(f_t) - C(a_t) + \delta EV_{t+1}(r_{t+1}, k_{t+1})$$

## b)Derive the Optimality Conditions

That is the FONCs w.r.t. the choice variable(s). Therefore

$$U_{f_t}(f_t) = \delta\beta EV_{t+1k_{t+1}}(r_{t+1}, k_{t+1}) \quad (1)$$

$$C_{a_t}(a_t) = \delta\gamma EV_{t+1k_{t+1}}(r_{t+1}, k_{t+1}) - \delta EV_{t+1r_{t+1}}(r_{t+1}, k_{t+1}) \quad (2)$$

## c)Apply the Envelope theorem

$$V_{r_t}(r_t, k_t) = \delta EV_{t+1r_{t+1}}(r_{t+1}, k_{t+1}) \quad (3)$$

$$V_{k_t}(r_t, k_t) = \delta EV_{t+1k_{t+1}}(r_{t+1}, k_{t+1}) \quad (4)$$

## d)Re-write the Optimality conditions and ET equations

From the optimality conditions

$$(1): \quad EV_{t+1k_{t+1}}(r_{t+1}, k_{t+1}) = \frac{U_{f_t}}{\delta\beta}$$

$$(2): \quad EV_{t+1r_{t+1}}(r_{t+1}, k_{t+1}) = \gamma EV_{t+1k_{t+1}}(r_{t+1}, k_{t+1}) - \frac{1}{\delta}C_{a_t}$$

$$EV_{t+1r_{t+1}}(r_{t+1}, k_{t+1}) = \gamma\frac{U_{f_t}(f_t)}{\beta\delta} - \frac{1}{\delta}C_{a_t}$$

We could re-write the ET(s) as

$$V_{r_t}(r_t, k_t) = \frac{\gamma}{\beta} U_{f_t}(f_t) - C_{a_t}(a_t) \quad (5)$$

$$V_{k_t}(r_t, k_t) = \frac{1}{\beta} U_{f_t}(f_t) \quad (6)$$

We could then re-write the optimality conditions as

$$U_{f_t}(f_t) = \delta U_{f_t}(f_{t+1}) \quad (7)$$

$$C_{a_t}(a_t) = \frac{\delta\gamma}{\beta} U_{f_t}(f_{t+1}) - \frac{\delta\gamma}{\beta} U_{f_t}(f_{t+1}) + \delta C_{a_t}(a_t) = \delta C_{a_t}(a_t) \quad (8)$$

**e) let $p_t = U_{f_t}(f_t)$ and $c_t = C_{a_t}(a_t)$. Show the optimality condition satisfies $p_t = \delta p_{t+1}$ and $c_t = \delta c_{t+1}$. What does this imply on the path of $a_t$ and $f_t$ given that $U(.)$ is concave, $C(.)$ is convex?**

Directly from (7) and (8) we can show that:

$$p_t = \delta p_{t+1}$$

$$c_t = \delta c_{t+1}$$

The path of the action variables are modelled by the optimality conditions. Hence, from

$$(1) \quad and \quad (2) \quad \geq 0$$

Since $U(.)$ is concave, by Definition $f_t \uparrow \rightarrow U_{f_t}(f_t) \downarrow$. Therefore, since $U_{f_t}(f_t) \uparrow$ then $f_t \downarrow$ in time and since $C(.)$ is convex, by Definition $k_t \geq r_t$ . Therefore, since $C_{a_t}(a_t) \uparrow$ then $a_t \uparrow$ in time.

**Good Luck!!!**
**Final Examination Details:**
**Date:** *** January 2020
**Time:****:00am - **:00noon
**Classroom:** TBA