Dynamic Optimization: Optimal Control

Introductory Mathematical Economics

David Ihekereleome Okorie

November 28th 2019

Outline

Outline

Outline

Variables and Functions

- s_t is a state variable at time t
- a_t is a control or (an) action variable at time t
- $U(s_t, a_t, t)$ is a single period reward or welfare function
- $s_{t+1} = g(s_t, a_t)$ or $s_{t+1} = s_t + f(s_t, a_t, t)$ is the transition equation for the state variable s_t
- t,T; $(T \leq \infty)$ is the time horizon which could be discrete or continuous
- s_0 is the initial state variable condition
- $s_T = k$ is a terminal condition when $T < \infty$
- $F(s_T)$ is the final period reward function

Outline

Outline

A Dynamic Optimization Problem

$$\underset{(s_t,a_t),0\leq t\leq T}{\text{argmax}} \quad U(s_0,a_0,0)+U(s_1,a_1,1)+U(s_2,a_2,2)+\dots$$

$$+\,U(s_{T-1},a_{T-1},T-1)+F(s_T)$$

$$s.t.\;\; s_1=s_0+f(s_0,a_0,0);\;\; s_2=s_1+f(s_1,a_1,1)\;\; s_3=s_2+f(s_2,a_2,2);\dots$$

$$;\, s_T=s_{T-1}+f(s_{T-1},a_{T-1},T-1).$$

Why don't we have a constraint for $t=T$?

$$Ł=[U(s_0,a_0,0)+\lambda_1(f(s_0,a_0,0)+s_0-s_1)]$$

$$+[U(s_1,a_1,1)+\lambda_2(f(s_1,a_1,1)+s_1-s_2)]$$

$$+[U(s_2,a_2,2)+\lambda_3(f(s_2,a_2,2)+s_2-s_3)]+\dots$$

$$+[U(s_{T-1},a_{T-1},T-1)+\lambda_T(f(s_{T-1},a_{T-1},T-1)+s_{T-1}-s_T)]$$

$$+F(s_T)$$

FONCS(a_t, $0 \leq t < T$):

$$\left.\begin{aligned}
\frac{\delta Ł}{\delta a_0} &= U_2 + \lambda_1 f_2 = 0 \\
\frac{\delta Ł}{\delta a_1} &= U_2 + \lambda_2 f_2 = 0 \\
\frac{\delta Ł}{\delta a_2} &= U_2 + \lambda_3 f_2 = 0 \\
&\vdots \\
\frac{\delta Ł}{\delta a_{T-1}} &= U_2 + \lambda_T f_2 = 0
\end{aligned}\right\} \frac{\delta Ł}{\delta a_t} = U_2 + \lambda_{t+1} f_2 = 0$$

$$\frac{\delta Ł}{\delta a_T} = 0 \;\; (why?)$$

FONCS(s_t, $0 \le t \le T$):

$$\left.\begin{aligned} \frac{\delta Ł}{\delta s_0} &= U_1 + \lambda_1 + \lambda_1 f_1 = 0 \\ \frac{\delta Ł}{\delta s_1} &= U_1 + \lambda_2 + \lambda_2 f_1 - \lambda_1 = 0 \\ \frac{\delta Ł}{\delta s_2} &= U_1 + \lambda_3 + \lambda_3 f_1 - \lambda_2 = 0 \\ &\vdots \\ \frac{\delta Ł}{\delta s_{T-1}} &= U_1 + \lambda_T + \lambda_T f_1 - \lambda_{T-1} = 0 \end{aligned}\right\} \frac{\delta Ł}{\delta s_t} = U_1 + \lambda_{t+1} + \lambda_{t+1} f_1 - \lambda_t = 0$$

$$\frac{\delta Ł}{\delta s_T} = -\lambda_T + F' = 0$$

FONCS(λ_t, $1 \leq t \leq T$):

$$\left.\begin{array}{c}
\frac{\delta Ł}{\delta\lambda_1} = s_0 + f(.) - s_1 = 0 \\
\frac{\delta Ł}{\delta\lambda_2} = s_1 + f(.) - s_2 = 0 \\
\\
\frac{\delta Ł}{\delta\lambda_3} = s_2 + f(.) - s_3 = 0 \\
\\
\vdots \\
\\
\frac{\delta Ł}{\delta\lambda_{T-1}} = s_{T-2} + f(.) - s_{T-1} = 0 \\
\\
\frac{\delta Ł}{\delta\lambda_T} = s_{T-1} + f(.) - s_T = 0
\end{array}\right\} \frac{\delta Ł}{\delta\lambda_{t+1}} = s_t + f(.) - s_{t+1} = 0$$

Definite Discrete Time Domain: Generalization

$$\underset{s_t,a_t}{\operatorname{argmax}} \sum_{t=0}^{T-1} U(s_t, a_t, t), \; +F(s_T)$$

$$s.t. \;\; s_{t+1} = g(s_t, a_t) \equiv s_{t+1} = s_t + f(s_t, a_t, t)$$

$$Ł = \sum_{t=0}^{T-1} [U(s_t, a_t, t) + \lambda_{t+1}(f(s_t, a_t, t) + s_t - s_{t+1})] + F(s_T)$$

$$FONCs: \quad \frac{\delta Ł}{\delta a_t} = \frac{\delta U(.)}{\delta a_t} + \lambda_{t+1} \frac{\delta f(.)}{\delta a_t} = 0$$

$$\frac{\delta Ł}{\delta s_t} = \frac{\delta U(.)}{\delta s_t} + \lambda_{t+1} + \lambda_{t+1} \frac{\delta f(.)}{\delta s_t} - \lambda_t = 0$$

$$\frac{\delta Ł}{\delta s_T} = -\lambda_T + \frac{dF(.)}{ds_T} = 0 \;\; \& \;\; \frac{\delta Ł}{\delta \lambda_{t+1}} = s_{t+1} = s_t + f(s_t, a_t, t) = 0$$

Continuous Time Domain: Generalization

$$\underset{s_t, a_t}{\text{argmax}} \int_{t=0}^{T-1} U(s(t), a(t), (t))dt + F(s(T))$$

$$s.t. \;\; s'(t) = f(s(t), a(t), (t))$$

$$Ł = \int_{t=0}^{T-1} [U(s(t), a(t), (t)) + \lambda(t)(f(s(t), a(t), t) - s'(t))]dt + F(s(T))$$

$$FONCs: \;\; \frac{\delta Ł}{\delta a(t)} = \frac{\delta U(.)}{\delta a(t)} + \lambda(t)\frac{\delta f(.)}{\delta a(t)} = 0$$

$$\frac{\delta Ł}{\delta s(t)} = \frac{\delta U(.)}{\delta s(t)} + \lambda(t)\frac{\delta f(.)}{\delta s(t)} - \lambda(t)\frac{\delta s'(t)}{\delta s(t)} = 0$$

$$\frac{\delta Ł}{\delta s(T)} = \frac{dF(.)}{ds(T)} = 0$$

$$\frac{\delta Ł}{\delta \lambda(t)} = f(s(t), a(t), (t)) - s'(t) = 0$$

$$s'(t) \equiv S_{t+1} - S_t$$

Outline

Outline

The Hamiltonian

Axiom

Given that the constraint is transitory [has $s'(t) \equiv s_{t+1} - s_t$], we can remodel using the Hamiltonian (removing the transitory component) for easy calculations.

Using discrete time domain

$$Ł = \sum_{t=0}^{T-1} [U(s_t, a_t, t) + \lambda_{t+1}(f(s_t, a_t, t) + s_t - s_{t+1})] + F(s_T)$$

$$Ł = \sum_{t=0}^{T-1} [U(s_t, a_t, t) + \lambda_{t+1} f(s_t, a_t, t) + \lambda_{t+1}[s_t - s_{t+1}]] + F(s_T)$$

$$Let \;\; H(a_t, s_t, t) = U(s_t, a_t, t) + \lambda_{t+1} f(s_t, a_t, t)$$

Therefore,

$$Ł = \sum_{t=0}^{T-1} [H(s_t, a_t, t) + \lambda_{t+1}[s_t - s_{t+1}]] + F(s_T)$$

$$FONCs: \quad \frac{\delta Ł}{\delta a_t} = \frac{\delta H(.)}{\delta a_t} = 0$$

$$\frac{\delta Ł}{\delta s_t} = \frac{\delta H(.)}{\delta s_t} + \lambda_{t+1} - \lambda_t = 0 \equiv -\frac{\delta H(.)}{\delta s_t} = \lambda_{t+1} - \lambda_t$$

$$\frac{\delta Ł}{\delta s_T} = -\lambda_T + \frac{dF(.)}{ds_T} = 0$$

$$\frac{\delta Ł}{\delta \lambda_{t+1}} = \frac{\delta H(.)}{\delta \lambda_{t+1}} + s_t - s_{t+1} = 0$$

In conclusion,

$$\frac{\delta H(.)}{\delta a_t} = 0 \rightarrow \frac{\delta U(.)}{\delta a_t} + \lambda_{t+1}\frac{\delta f(.)}{\delta a_t}$$

$$-\frac{\delta H(.)}{\delta s_t} = \lambda_{t+1} - \lambda_t \rightarrow -\frac{\delta U(.)}{\delta s_t} - \lambda_{t+1}\frac{\delta f(.)}{\delta s_t} = \lambda_{t+1} - \lambda_t$$

$$\frac{dF(.)}{ds_T} = \lambda_T \rightarrow \frac{dF(.)}{ds_T} = \lambda_T$$

$$\frac{\delta H(.)}{\delta \lambda_{t+1}} = s_{t+1} - s_t \rightarrow f(s_t, a_t, t) = s_{t+1} - s_t$$

Similar, applying the Hamiltonian approach to continuous time:

Continuous time domain Hamiltonian

$$Recall \quad \underset{s_t, a_t}{\text{argmax}} \int_{t=0}^{T-1} U(s(t), a(t), (t)) + F(s(T))$$

$$s.t. \;\; s'(t) = f(s(t), a(t), (t))$$

We therefore define the Hamiltonian:

$$H(a(t), s(t), t) = U(s(t), a(t), t) + \lambda(t) f(s(t), a(t), t)$$

$$FONCs: \frac{\delta H(.)}{\delta a(t)} = 0 \rightarrow \frac{\delta U(.)}{\delta a(t)} + \lambda(t) \frac{\delta f(.)}{\delta a(t)} = 0$$

$$-\frac{\delta H(.)}{\delta s(t)} = \lambda'(t) \rightarrow -\frac{\delta U(.)}{\delta s(t)} - \lambda(t) \frac{\delta f(.)}{\delta s(t)} = \lambda'(t)$$

$$-\lambda_T + \frac{dF(.)}{ds(T)} = 0 \rightarrow \frac{dF(.)}{ds(T)} = \lambda(T)$$

$$\frac{\delta H(.)}{\delta \lambda(t)} = s'(t) \rightarrow f(s(t), a(t), t) = s'(t)$$

Outline

Outline

The present value of Hamiltonian

The Idea

Discount rates (δ) and Discount factors $(1+\delta)^{-t}$ for discrete time & $e^{-\delta t}$ for continuous time, where $t \leq T$, are used to discount the Hamiltonian to a present value Hamiltonian such that the Hamiltonian doesn't depend **explicitly** but **implicitly** on time (t) through the discount factor.

The Transformation

$$\left.\begin{array}{c} H(a_t, s_t, t) = U(s_t, a_t, t) + \lambda_{t+1} f(s_t, a_t, t) \\ H(a, s) = U(s, a)(1+\delta)^{-t} + \lambda f(s, a) \end{array}\right\} \textit{Discrete Time Horizon}$$

$$\left.\begin{array}{c} H(a(t), s(t), t) = U(s(t), a(t), t) + \lambda(t) f(s(t), a(t), t) \\ H(a, s) = U(s, a) e^{-\delta t} + \lambda f(s, a) \end{array}\right\} \textit{Continuous}$$

These are the Hamiltonian for discrete and continuous horizon and their present value Hamiltonian . Note: $F(s(T))$ becomes $F(s(T))e^{-\delta T}$.

The FONCs of a P.R.F.V.H

Applying the Hamiltonian Approach:

$$H(a, s) = U(s, a)e^{-\delta t} + \lambda f(s, a)$$

FONCs of PVH

$$FONC(a): \;\; U_a(s, a)e^{-\delta t} + \lambda f_a(s, a) = 0$$

$$FONC(s): \;\; -U_s(s, a)e^{-\delta t} - \lambda f_s(s, a) = \lambda'$$

$$FONC(\lambda): \;\; f(s, a) = s'$$

Outline

Outline

Current Value Hamiltonian

The idea here is to explicitly make the Hamiltonian independent of time (t). However, the current reward function value of hamiltonian is still **implicitly** dependent on time(t). The later will be clear soon, while getting the FONCs of a current value hamiltonian.

The Transformation

$$H(a,s) = U(s,a)e^{-\delta t} + \lambda f(s,a) \ldots PVH$$

$$\hat{H}(s,a) = H(a,s) \times e^{\delta t} = U(s,a) + \lambda e^{\delta t} f(s,a)$$

$$Let \;\; \mu = \lambda e^{\delta t}$$

$$\hat{H}(s,a) = U(s,a) + \mu f(s,a) \ldots CVH$$

FONCs of CVH

$$FONC(\mu): \quad f(s,a) = s'$$
$$FONC(a): \quad U_a(s,a) + \mu f_a(s,a) = 0$$
$$FONC(s): \quad -U_s(s,a) - \mu f_s(s,a) = \lambda' e^{\delta t}$$

Note: The FONCs of CVH $= e^{\delta t} PVH$. However, we do not have λ in CVH, so what is $\lambda' e^{\delta t}$?

λ'

$$\mu = \lambda e^{\delta t}$$

Recall, λ **implicitly** depends on time, so $\lambda' = \frac{d\lambda}{dt}$ exits. Clearly, we can write

$$\mu' = \lambda \delta e^{\delta t} + \lambda' e^{\delta t}$$
$$\lambda' e^{\delta t} = \mu' - \delta \lambda e^{\delta t}$$
$$\lambda' e^{\delta t} = \mu' - \delta \mu$$

Clearly, we can rewrite the FONCs of a CVH as

$$\frac{\delta \hat{H}}{\delta \mu} = s'$$

$$\frac{\delta \hat{H}}{\delta a} = 0$$

$$-\frac{\delta \hat{H}}{\delta s} = \mu' - \delta \mu$$

Outline

Outline

Fishery Problem

- U(s(t),a(t).t) = ln(a(t))
- $s'(t) = f(s(t), a(t))$
- $f(s(t), a(t)) = s(t)[1 - \frac{s(t)}{s(0)}] - a(t)$
- $\delta = 0.2$
- s(0) = 100
- a(t) is fish harvest at time (t); s(t) is the stock of fish at time (t); & U(s(t),a(t)) is the reward function at time (t)

Tasks:

1. Solve for the steady state values $\mu^\star$, $a^\star$, and $s^\star$
2. Conduct a stability analysis
3. What type of local or global steady state exists in this problem

Solving for the steady state values

$$H = ln(a(t))e^{-0.2t} + \lambda[s(t)(1 - \frac{s(t)}{100}) - a(t)]$$

$$\hat{H} = ln(a) + \mu[s(1 - \frac{s}{100}) - a]$$

are the PVH and CVH respectively.

FONCs of CVH

$$\frac{1}{a} - \mu = 0 \quad eqn(1)$$

$$-\mu(1 - \frac{s}{50}) = \mu' - 0.2\mu \quad eqn(2)$$

$$s[1 - \frac{s}{100}] - a = s' \quad eqn(3)$$

From eqn.(2)

$$s = 40 + 50\frac{\mu'}{\mu} \rightarrow s^{\star} = 40|\mu' = 0$$

From eqn.(3)

$$a^{\star} = s^{\star}[1 - \frac{s^{\star}}{100}]|s' = 0 \rightarrow a^{\star} = 24$$

From eqn.(1)

$$\mu^{\star} = \frac{1}{a^{\star}} \rightarrow \frac{1}{24}$$

Therefore,

$$s^{\star} = 40, \quad a^{\star} = 24 \quad \mu^{\star} = \frac{1}{24}$$

Conducting a stability analysis

Recall our system of equations:

$$\frac{1}{a} - \mu = 0 \quad eqn(1)$$

$$-\mu(1 - \frac{s}{50}) = \mu' - 0.2\mu \quad eqn(2)$$

$$s(t)[1 - \frac{s}{100}] - a = s' \quad eqn(3)$$

We need a DTE system of equations for the action and state variables.

Combine (1) and (2)

$$-a^{-1}[1 - s50^{-1}] = -a^{-2}a' - 0.2a^{-1}$$

$$a' = [0.8 - \frac{s}{50}]a$$

$$s' = s[1 - \frac{s}{100}] - a \quad eqn(3)$$

Isocline of $s' = 0$

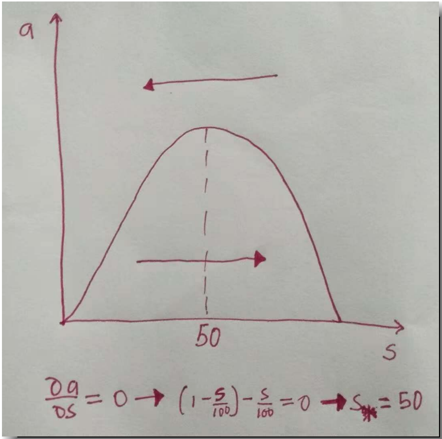

Figure 1: s' Isocline

From $s' = s[1 - \frac{s}{100}] - a$ we set $s' = 0$ and plot $a = s[1 - \frac{s}{100}]$ using the fact that a is concave in s (concave parabola i.e. negative quadratic coefficient).Consider what happens to s' when a increases and decreases.

1.)From $s' = 0$ upwards, a increases $\rightarrow$ s' decreases from $s' = s[1 - \frac{s}{100}] - a$

2.)From $s' = 0$ downwards, a decreases and s' increases.

Isocline of $a' = 0$

From $a' = [0.8 - \frac{s}{50}]a$ we set $a' = 0$ and plot $s = 40$. Consider what happens to a' when s increases and decreases.

1.)From $a' = 0$ rightward, s increases and a' decreases

2.)From $a' = 0$ leftward, s decreases and a' increases

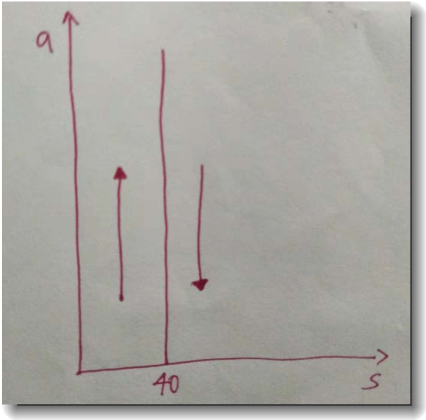

Figure 2: a' Isocline

Combining the Isoclines

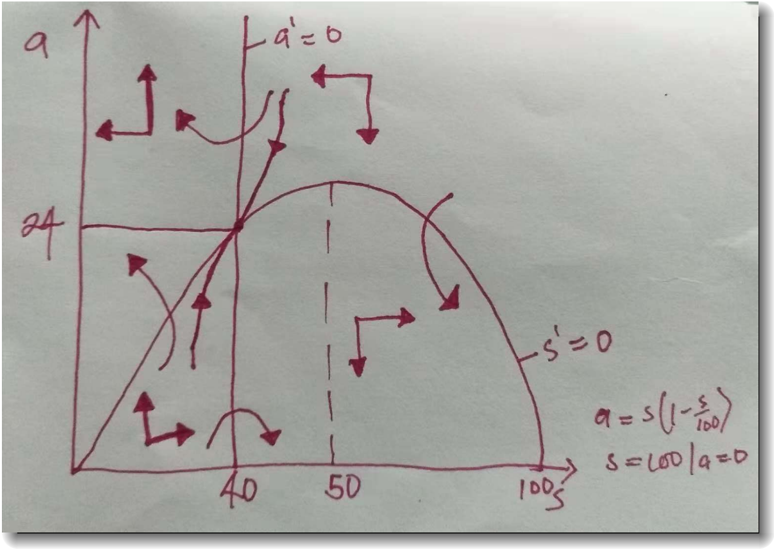

Figure 3: Phase Diagram

Type of Stability

Recall

$$a' = [0.8 - \frac{s}{50}]a \rightarrow G(s,a)$$

$$s' = s[1 - \frac{s}{100}] - a \quad \rightarrow F(s,a)$$

$$s^\star = 40, \quad a^\star = 24$$

To calculate the stability form, we need to examine the eigenvalues using the Jacobian matrix.

$$J = \begin{bmatrix} F_s(s,a) & F_a(s,a) \\ G_s(s,a) & G_a(s,a) \end{bmatrix} |s^\star, a^\star = \begin{bmatrix} 0.2 & -1 \\ -0.48 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.2 - \lambda & -1 \\ -0.48 & -\lambda \end{bmatrix} \quad \rightarrow \quad \lambda = 0.8 \ \ or \quad -0.6$$

Therefore, we have a local saddle point (since the eigenvalues are positive and negative).